Lattices and the complexity of maximum constraint satisfaction

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Constraints

- D – a finite set with $|D| > 1$;
- $R_D^{(m)} = \{f \mid f : D^m \rightarrow \{0, 1\}\}$, $R_D = \bigcup_{m=1}^{\infty} R_D^{(m)}$.

Definition 1 *A constraint over a set of variables* $V = \{x_1, x_2, \ldots, x_n\}$ *is an expression of the form* $f(\mathbf{x})$ *where*

- $f \in R_D^{(m)}$ *is the constraint function,*
- $\mathbf{x} = (x_{i_1}, \ldots, x_{i_m})$ *the constraint scope.*

The constraint $f(\mathbf{x})$ *is said to be satisfied on a tuple* $\mathbf{a} = (a_{i_1}, \ldots, a_{i_m}) \in D^m$ *if* $f(\mathbf{a}) = 1$.

Maximum constraint satisfaction problem

Max CSP

Instance: A collection $f_1(\mathbf{x}_1), \ldots, f_q(\mathbf{x}_q)$ of constraints over $V = \{x_1, \ldots, x_n\}$;
each constraint $f_i(\mathbf{x}_i)$ has a weight $w_i \in \mathbb{Z}^+$.

Goal: Find an assignment $\phi : V \to D$ that maximises the total weight of satisfied constraints; in other words, maximise the function $f : D^n \to \mathbb{Z}^+$, defined by

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{q} w_i \cdot f_i(\mathbf{x}_i).$$

Parameterisation of MAX CSP

For a finite set $\mathcal{F} \subseteq R_D$,
MAX CSP($\mathcal{F}$) consists of all MAX CSP instances in which all constraint functions f_i belong to $\mathcal{F}$.

Example 1 *Let $D = \{0, 1\}$ and $\mathcal{F} = \{neq\}$ where $neq(x, y) = 1$ if $x \neq y$ and $neq(x, y) = 0$ otherwise. Then* MAX CSP($\mathcal{F}$) *is precisely* MAX CUT.

Indeed, for a graph $G = (V, E)$ with $V = \{x_1, \ldots, x_n\}$, computing maximum cut is the same as maximising

$$f(x_1, \ldots, x_n) = \sum_{e=(x_i,x_j)\in E} w_e \cdot neq(x_i, x_j).$$

Classification problem

Problem 1 *Characterise (assuming that* **P** $\neq$ **NP***) sets* $\mathcal{F}$ *such that*

- Max CSP($\mathcal{F}$) *is tractable (i.e., in* **PO***)*
- Max CSP($\mathcal{F}$) *is* **NP***-hard.*

Example 2 *The problem* Max CSP($\{neq\}$) *(*Max Cut*) from the previous slide is* **NP***-hard.*

Supermodularity on lattices

Definition 2 *Let $\mathcal{L}$ be a lattice on a finite set D. A function $f : D^n \to \mathbb{Q}$ is called supermodular on $\mathcal{L}$ if*

$$f(\mathbf{a}) + f(\mathbf{b}) \leq f(\mathbf{a} \sqcup \mathbf{b}) + f(\mathbf{a} \sqcap \mathbf{b}) \text{ for all } \mathbf{a}, \mathbf{b} \in D^n.$$

Problem 2 *Fix a finite lattice $\mathcal{L}$ and let $\mathrm{SFM}(\mathcal{L})$ be the problem of maximising a given n-ary supermodular function f on $\mathcal{L}$. Is there an algorithm solving $\mathrm{SFM}(\mathcal{L})$ in polynomial time in n and FE?*

Theorem 1 (Schrijver'00, Iwata et al.'01)
$\mathrm{SFM}(\mathcal{L})$ *is tractable for any distributive lattice $\mathcal{L}$.*

From SFM($\mathcal{L}$) to Max CSP($\mathcal{F}$)

Assume that

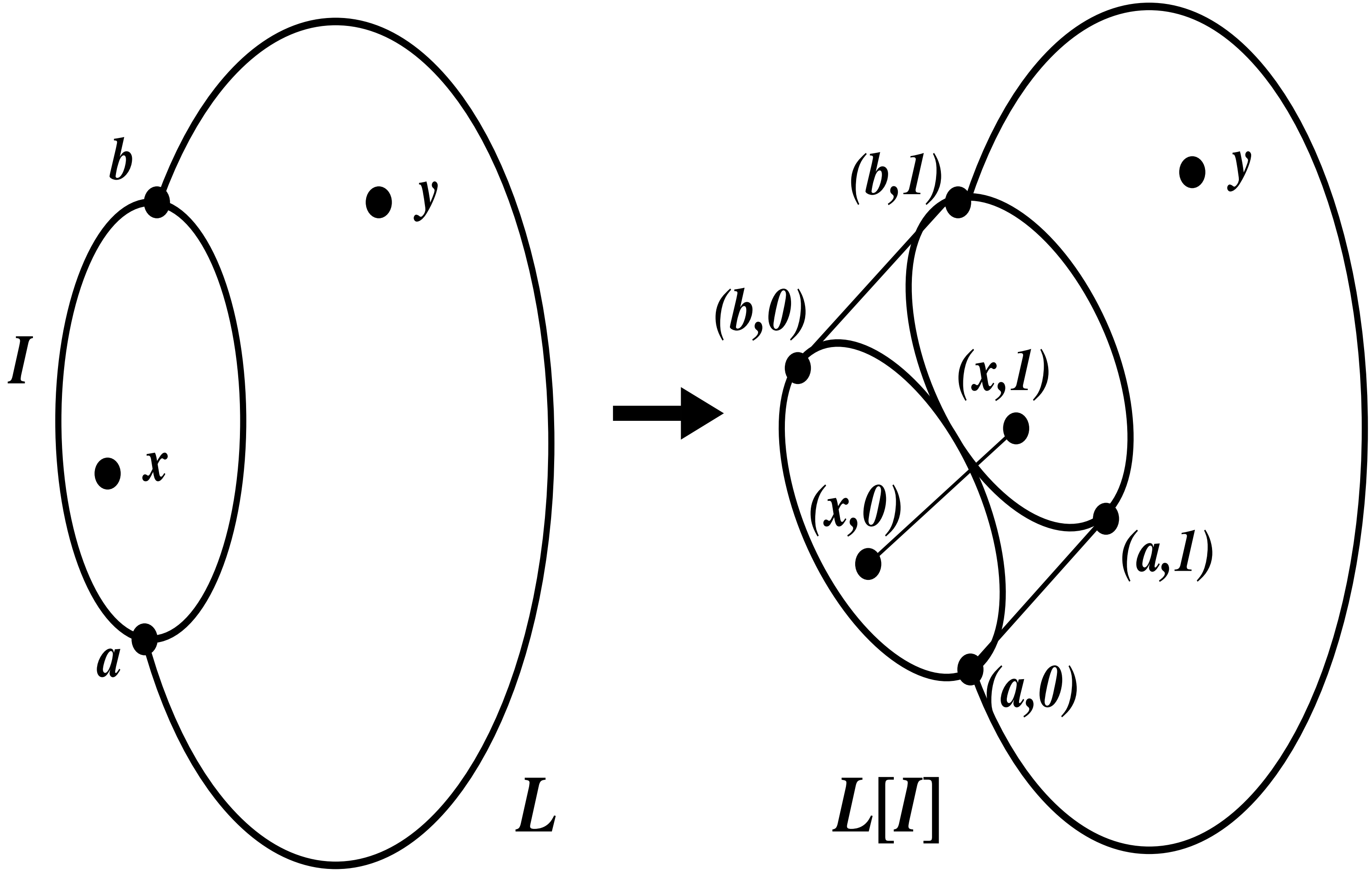
- $\mathcal{F}$ consists of supermodular 0-1 functions on $\mathcal{L}$, and
- SFM($\mathcal{L}$) is tractable.

Then

- $f(x_1, \ldots, x_n) = \sum_{i=1}^{q} w_i \cdot f_i(\mathbf{x}_i)$ is supermodular on $\mathcal{L}$ and, moreover, FE is linear in q,
- so the algorithm for SFM($\mathcal{L}$) solves Max CSP($\mathcal{F}$) in polynomial time.

Corollary 1 Max CSP($\mathcal{F}$) *is tractable if* $\mathcal{F}$ *consists of supermodular 0-1 functions on a finite distributive lattice.*

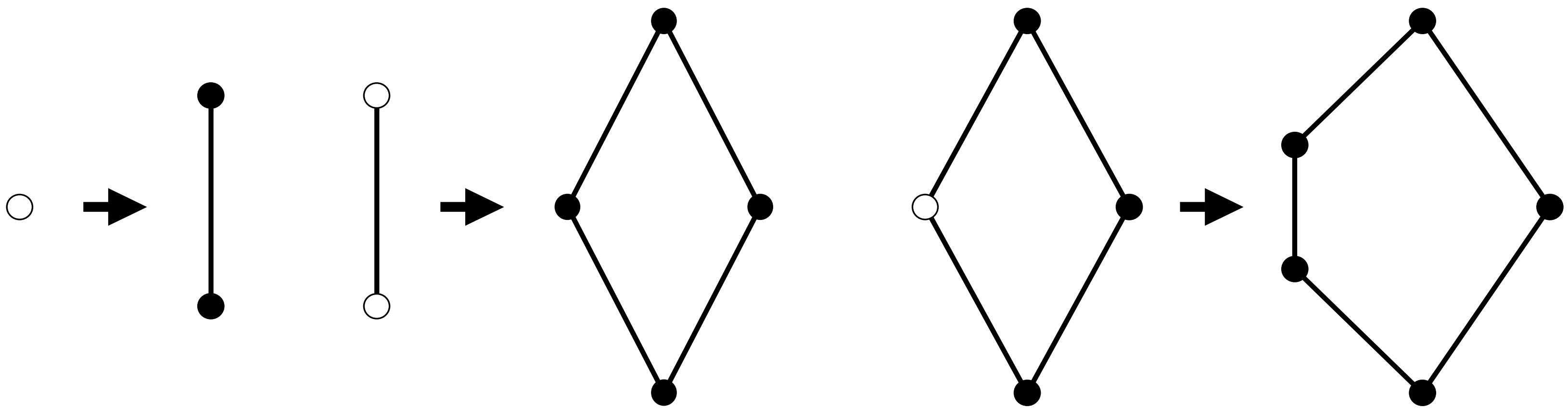
Interval-doubling construction



Finite bounded lattices

Definition 3 *A finite bounded lattice is a lattice that can be obtained from the one-element lattice by successive doubling of intervals.*

Example 3 *The pentagon N_5 is a bounded lattice.*



A tractability result

Theorem 2 (AK, Larose '06)
For any fixed finite bounded lattice $\mathcal{L}$, the problem $\mathrm{SFM}(\mathcal{L})$ *can be solved in polynomial time in n and FE.*

Facts about the class of finite bounded lattices:

- contains all finite distributive lattices
- pseudovariety (closed under H, S, P_{fin})
- satisfies no non-trivial lattice identity
- the smallest lattice not in the class is the diamond M_3

Corollary 2 MAX CSP$(\mathcal{F})$ *is tractable if $\mathcal{F}$ consists of supermodular 0-1 functions on a finite bounded lattice.*

Supermodular 0-1 functions

Let $\mathcal{L}$ be a lattice on D and f an m-ary 0-1 function on D.
Set $S_f = \{\mathbf{x} \in D^m \mid f(\mathbf{x}) = 1\}$.
Then the supermodularity (on $\mathcal{L}$) condition for f

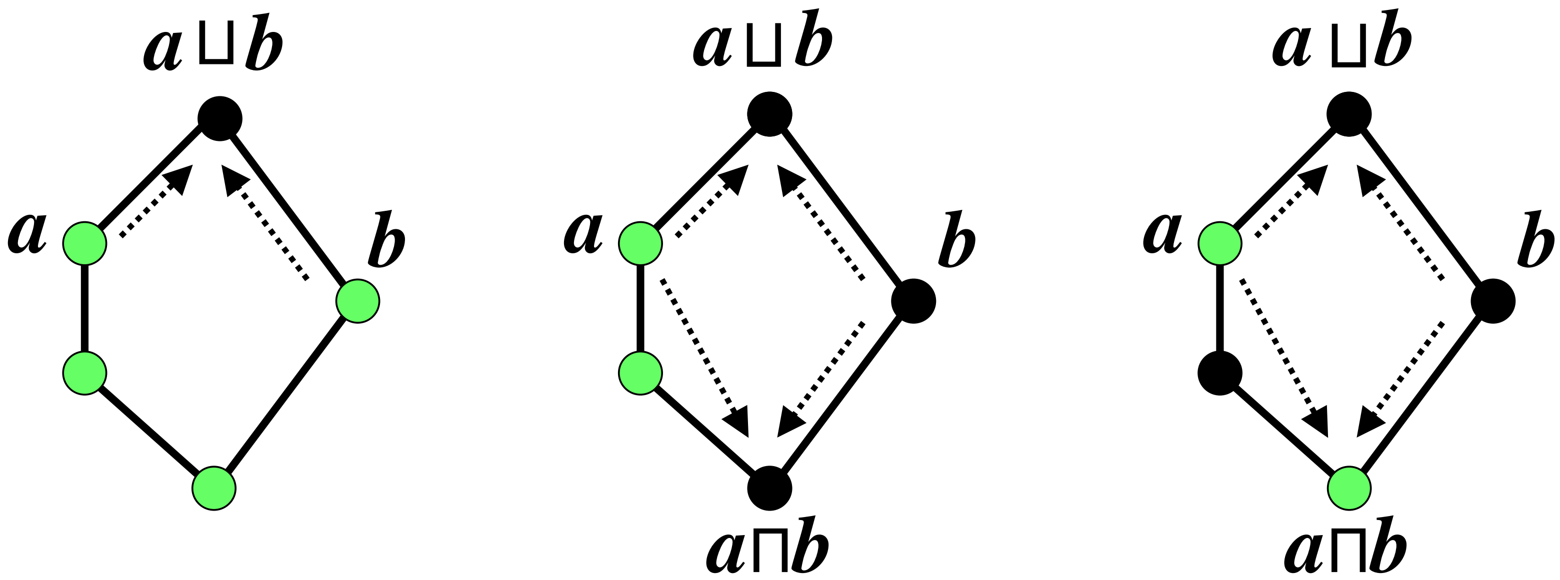
$$f(\mathbf{a}) + f(\mathbf{b}) \leq f(\mathbf{a} \sqcup \mathbf{b}) + f(\mathbf{a} \sqcap \mathbf{b})$$

can be expressed as the following

1. $\mathbf{a}, \mathbf{b} \in S_f \Rightarrow \mathbf{a} \sqcup \mathbf{b}, \mathbf{a} \sqcap \mathbf{b} \in S_f$ where $\sqcup$ and $\sqcap$ act component-wise (i.e., S_f is a sublattice of $\mathcal{L}^m$), and
2. $\mathbf{a} \in S_f, \mathbf{b} \notin S_f \Rightarrow \{\mathbf{a} \sqcup \mathbf{b}, \mathbf{a} \sqcap \mathbf{b}\} \cap S_f \neq \emptyset$, i.e., S_f is a sort of "semi-ideal, semi-filter" of $\mathcal{L}^m$.

Examples

- *elements in S_f*



Not a sublattice, so not supermod

Sublattice, but not supermod

Sublattice, supermod

2-monotone functions on lattices

Definition 4 *A function $f \in R_D^{(m)}$ is called 2-monotone on a lattice $\mathcal{L}$ if S_f is either an ideal, or a filter, or a union of an ideal and a filter in $\mathcal{L}^m$.*

Fact 1 *f is 2-monotone on $\mathcal{L} \Rightarrow f$ is supermodular on $\mathcal{L}$.*

Theorem 3 (Cohen, Cooper, Jeavons, AK '05)
Let $\mathcal{L}$ be a lattice on D, and $\mathcal{F} \subseteq R_D$ consist of 2-monotone functions on $\mathcal{L}$. Then Max CSP($\mathcal{F}$) *is in* **PO**.

- For each finite lattice $\mathcal{L}$, there exists Max CSP($\mathcal{F}$) whose tractability can (now) be explained only by supermodularity on this lattice $\mathcal{L}$.

Classification for small domains

For $|D| = 2$, the complexity of MAX CSP($\mathcal{F}$) was classified by Creignou (1995) without using supermodularity.

Theorem 4 (Jonsson, Klasson, AK '06)
Let $|D| \leq 3$ and let $\mathcal{F} \subseteq R_D$ be a core.

- *If there is a chain $\mathcal{L}$ on D such that all functions in $\mathcal{F}$ are supermodular on $\mathcal{L}$ then* MAX CSP($\mathcal{F}$) *is in* **PO**.
- *Otherwise,* MAX CSP($\mathcal{F}$) *is* **NP***-hard.*

NB. This classification result is a dichotomy theorem, it says problems are either easy or as hard as can be.

Classification with "constants"

For $d \in D$, let $u_d(x) = 1$ iff $x = d$. Let $\mathcal{C}_D = \{u_d \mid d \in D\}$.

Having $\mathcal{C}_D \subseteq \mathcal{F}$ is equivalent to allowing constraints of the form $w_i \cdot u_d(x)$, specifying how much you want that $x = d$.

Theorem 5 (Deineko, Jonsson, Klasson, AK '06)
Let D be any finite set and let $\mathcal{C}_D \subseteq \mathcal{F} \subseteq R_D$.

- *If there is a chain $\mathcal{L}$ on D such that all functions in $\mathcal{F}$ are supermodular on $\mathcal{L}$ then* Max CSP($\mathcal{F}$) *is in* **PO**.
- *Otherwise,* Max CSP($\mathcal{F}$) *is* **NP***-hard.*

NB. It is easy to check that every u_d is supermodular on a lattice iff the lattice is a chain.