Free objects in the class of power left and right idempotent groupoids

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1 Preliminaries

Let $\mathcal{V}$ be any variety of groupoids.
A groupoid $\boldsymbol{G}$ is called an *n-$\mathcal{V}$-groupoid* iff

$$(\forall a_1, a_2, \ldots, a_n \in G) \ \ \langle a_1, a_2, \ldots, a_n \rangle \in \mathcal{V}.$$

The class of all n-$\mathcal{V}$-groupoids is denoted by n-$\mathcal{V}$.

1-$\mathcal{V}$-groupoids are called *power $\mathcal{V}$-groupoids*; $p\mathcal{V}$.

$\mathcal{V} \subseteq n$-$\mathcal{V}$ and $(n+k)$-$\mathcal{V} \subseteq n$-$\mathcal{V}$, for any $n, k \geq 1$.
If the variety has the axiomatic rank n, then n-$\mathcal{V}$=$\mathcal{V}$.

$\mathcal{U} = Var[x^2y^2 \approx xy]$:

[1] Čupona G., Celakoski N., *On groupoids with the identity* $x^2y^2 = xy$, MANU (1997), 5–15

$p\mathcal{U}$-groupoids:
groupoids $\boldsymbol{G}$ such that $(\forall a \in G)\ \langle a \rangle \in \mathcal{U}$

B – arbitrary nonempty set

$\boldsymbol{F} = (F, \cdot)$ – the set of all groupoid terms over B in the signature $\cdot$. The terms are denoted by $t, u, v, w, \ldots$

$\boldsymbol{F} = (F, \cdot)$ is the absolutely free groupoid with the free basis B, where the operation is defined by $(u, v) \mapsto uv$.

For each $v \in F$, we define the *length* $\mid v \mid$ of v and the *set of subterms* $P(v)$ of v by:

$$\mid b \mid = 1, \ \mid tu \mid = \mid t \mid + \mid u \mid \tag{1.1}$$

$$P(b) = \{b\}, \ P(tu) = \{tu\} \cup P(t) \cup P(u), \tag{1.2}$$

for each $b \in B$ and $t, u \in F$.

$\boldsymbol{E} = (E, \cdot)$ – the absolutely free groupoid with one-element basis $\{e\}$; the elements of $\boldsymbol{E}$ are called *groupoid powers* and are denoted by $f, g, h, \ldots$.

For any groupoid $\boldsymbol{G} = (G, \cdot)$, each element $f \in E$ induces a transformation $f^{\boldsymbol{G}} : G \longrightarrow G$ (called an *interpretation* of f in $\boldsymbol{G}$) defined by:

$$e^{\boldsymbol{G}}(x) = x, \quad (gh)^{\boldsymbol{G}}(x) = g^{\boldsymbol{G}}(x)h^{\boldsymbol{G}}(x) \tag{1.3}$$

for any $g, h \in E$ and $x \in G$.

2 Construction of free objects in the variety $p\mathcal{U}$

It is shown in [1] that the axiom $x^2y^2 \approx xy$ is equivalent with the system of axioms $x^2y \approx xy$, $xy^2 \approx xy$; Also, if $\boldsymbol{G} \in \mathcal{U}$, then $a \in G$ is a square iff a is an idempotent.

Theorem 2.1. $\boldsymbol{G} \in p\mathcal{U}$ *iff*

$$(\forall x \in G)(\forall f \in E \setminus \{e\})\ f(x) = x^2.$$

Corollary 2.1. $\boldsymbol{G} \in p\mathcal{U}$ *iff*

$$(f(x))^2(g(x))^2 = f(x)g(x) \tag{2.1}$$

for any $x \in G$ *and* $f, g \in E$.

Corollary 2.2. *The class of* $p\mathcal{U}$*-groupoids is a variety defined by the identities*

$$x^2 \approx x^2x \approx xx^2 \approx x^2x^2. \tag{2.2}$$

An element $c \in G$ is said to be *primitive* in $\boldsymbol{G}$ iff

$$(\forall a \in G)(\forall f \in E \setminus \{e\}) \;\; c \neq f(a).$$

Lemma 2.1. *For any* $v \in \boldsymbol{F}$ *there is a uniquely determined primitive element* $u \in F$ *and uniquely determined* $f \in E \setminus \{e\}$ *such that* $v = f(u)$.

In that case we say that u is a *base* of v, f is the *power* of v, $|\, f \,|$ the *exponent* of v, and denote it by $\underline{v} = u$, $v^{\sim} = f$ and $|\, v^{\sim} \,|$, respectively.

Lemma 2.2. *Let* $v, w \in F$.

a) If v *and* w *have different bases, then* vw *is primitive element in* $\boldsymbol{F}$, *i.e.* ($\underline{v} \neq \underline{w} \Rightarrow vw = \underline{vw}$).

b) The elements v, w *have the same base* t *iff* t *is a base of* vw *(i.e.* $t = \underline{vw}$*) and the power of* vw *equals the product of the power of* v *and the power of* w *(i.e.* $\underline{v} = \underline{w} = t \;\Leftrightarrow\; t = \underline{vw} \;\&\; (vw)^{\sim} = v^{\sim}w^{\sim}$*).*

Define a carrier R of a free object in the variety $p\mathcal{U}$ as follows:

$$R = \{t \in F : \ (\forall u \in P(t)) \, | \, u^{\sim} \, | \leq 2\}.$$

Define an operation $*$ on R by:

$$t, u \in R \Rightarrow t{*}u = \begin{cases} tu, & \text{if} \quad tu \in R \\ x^2, & \text{if} \quad \underline{t} = \underline{u} = x, |\, t^{\sim} \,| + |\, u^{\sim} \,| \geq 3. \end{cases}$$

1°. $\boldsymbol{R} = (R, *)$ is a groupoid, B is the set of primes in $\boldsymbol{R}$ and generates $\boldsymbol{R}$.

Let $t \in R$, $f \in E$. We define $f_*(t)$ as follows:

$$e_*(t) = t, \ (fg)_*(t) = f_*(t) * g_*(t). \qquad (2.3)$$

2°. Let $t \in R$ and $f \in E \setminus \{e\}$.

a) If t is not a square in $\boldsymbol{F}$, then $f_*(t) = t^2$, and, specially, $t * t = t^2$.

b) If t is a square in $\boldsymbol{F}$, i.e. $t = x^2$, where x is a primitive element in $\boldsymbol{F}$, then $f_*(t) = t$, and, specially, $t * t = t$.

3°. $\boldsymbol{R} \in p\mathcal{U}$.
4°. $\boldsymbol{R}$ has the universal mapping property for $p\mathcal{U}$ over B.

Theorem 2.2. $\boldsymbol{R} = (R, *)$ *is a free groupoid in* $p\mathcal{U}$ *with a free basis* B.

The class of free objects in $p\mathcal{U}$ will be denoted by $p\mathcal{U}_{fr}$.

Note that, if $a, b \in B$ are distinct elements in B, then $a * (b * b) = a * b^2 = ab^2 \neq ab = a * b$, and therefore: if $|\, B \,| \geq 2$, then $\boldsymbol{R} \notin \mathcal{U}$.

Proposition 2.1. *a) For any $x \in R$, $x*x$ is an idempotent in $\boldsymbol{R}$.*
b) $t \in R$ is an idempotent in $\boldsymbol{R}$ iff t is a square in $\boldsymbol{R}$.
c) If $t \in R$ is an idempotent in $\boldsymbol{R}$, then there is a unique nonidempotent $x \in R$ (i.e. $x \neq x * x$) such that $t = x * x$.
d) $t \in R$ is primitive in $\boldsymbol{R}$ iff t is primitive in $\boldsymbol{F}$.

Proposition 2.2. *Let I be the set of all idempotents in $\boldsymbol{R}$ and N be the set of all nonidempotents in $\boldsymbol{R}$. If $z \in R$ is a nonidempotent and z is not prime in $\boldsymbol{R}$, then one of the following cases is possible:*
1) $z = \alpha * \beta$, $\alpha, \beta \in I$, $\alpha \neq \beta$;
2) $z = x * \alpha$, $\alpha \in I$, $x \in N$;
3) $z = \alpha * x$, $\alpha \in I$, $x \in N$;
4) $z = x * y$, $x, y \in N$, $x \neq y$.

Proposition 2.3. *For any $t \in R \setminus B$ there is exactly one pair $(u, v) \in R^2$, such that* $\quad t = uv = u * v$.

(We say that (u, v) is the *pair of divisors* of t in $\boldsymbol{R}$. In this case: $u = v$ iff $u^2 \in R$ (iff u is not a square); then we say that u is the *divisor* of t).

Remark. The proposition 2.3. does not exclude existence of distinct pairs $(u_1, v_1), (u_2, v_2) \in R^2$, such that $u_1 * v_1 = u_2 * v_2$.

3 Injective objects in $p\mathcal{U}$

We say that a groupoid $\boldsymbol{H} = (H, \cdot)$ is *injective* in $p\mathcal{U}$ (i.e. *$p\mathcal{U}$-injective*) iff

0) $\boldsymbol{H} \in p\mathcal{U}$

1) If $a \in H$ is an idempotent, then there is a unique nonidempotent $c \in H$, such that $a = c^2$ and the equality $a = xy$ holds iff $\{x, y\} \subseteq \{c, c^2\}$.
(In that case we say that c is the *divisor* of a or c is the *base* of a.)

2) If $a \in H$ is an nonidempotent and is not prime in $\boldsymbol{H}$, then there is a unique pair $(c, d) \in H^2$, such that $a = cd$ and $\underline{c} \neq \underline{d}$.
(Note that c, d could be both idempotents; one idempotent and the other nonidempotent; both nonidempotents.)

The class of all $p\mathcal{U}$-injective groupoids will be denoted by $p\mathcal{U}_{inj}$.

Proposition 3.1. *Every $p\mathcal{U}$-free object is $p\mathcal{U}$-injective, i.e. $p\mathcal{U}_{fr} \subseteq p\mathcal{U}_{inj}$.*

If $\boldsymbol{H} = (H, \cdot) \in p\mathcal{U}_{inj}$ and $a \in H$, then the subgroupoid $Q = \{a^2\}$ of $\boldsymbol{H}$ is not $p\mathcal{U}$-injective.

Proposition 3.2. *Neither of the classes $p\mathcal{U}_{fr}, p\mathcal{U}_{inj}$ is hereditary.*

Theorem 3.1. (Bruck Theorem for the variety $p\mathcal{U}$) *A groupoid $\boldsymbol{H} \in p\mathcal{U}$ is $p\mathcal{U}$-free iff the following conditions are satisfied:*

(i) $\boldsymbol{H}$ is $p\mathcal{U}$-injective.

(ii) The set P of primes in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$.

A construction of $p\mathcal{U}$-injective groupoid that is not $p\mathcal{U}$-free.

Let N be an infinite set, I a set equivalent and disjoint with N, $H = N \cup I$ and $\varphi : N \to I$ a bijection. We define the set $D = \{(x, y) : x, y \in N \cup I, x \neq y\}$. Since N is infinite it follows that the sets N, I and D are equivalent. This implies that there is an injection $\psi : D \to N$.

Define an operation "$\cdot$" in $H = N \cup I$ by:

$$n \cdot n = \varphi(n), \quad i \cdot i = i, \quad x \cdot y = \psi(x, y),$$

for any $n \in N$, $i \in I$, $(x, y) \in D$.

We obtain that $\boldsymbol{H} = (H, \cdot)$ is a groupoid that is $p\mathcal{U}$-injective. When ψ is a bijection, then there are no prime elements in $\boldsymbol{H}$. So, $\boldsymbol{H}$ is not $p\mathcal{U}$-free. Thus:

Proposition 3.3. *The class of $p\mathcal{U}$-free groupoids is a proper subclass of the class of $p\mathcal{U}$-injective groupoids, i.e. $p\mathcal{U}_{fr} \subset p\mathcal{U}_{inj}$.*