Complexity of Variety Membership for nonfinitely based finite semigroups

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Var-Memb problem

Given a finite semigroup S, the Variety Membership problem for S is the following combinatorial decision problem (denoted by VAR-MEMB(S)):

INSTANCE: a finite semigroup S';

QUESTION: Does S' belong to the variety generated by S?

The decidability of this problem for finite universal algebras can be obtained from Tarski's HSP-theorem and this was shown in the work

J. KALICKI. On comparison of finite algebras, 1952.

Complexity of Var-Memb

The computational complexity of the Variety Membership problem for finite universal algebras has been investigated lately by BERGMAN C., SLUTZKI G., SZÉKELY Z., KOZIK M.

And the tight upper bound of the problem's complexity is 2EXPTIME.

The first examples of finite semigroups S for which the problem VAR-MEMB(S) is NP-hard have been constructed in

M. JACKSON, R. MCKENZIE. Interpreting graph colorability in finite semigroups, 2006.

Equational language

The Variety Membership problem VAR-MEMB(S) can be formulated in an equational language:

INSTANCE: a finite semigroup S';

QUESTION: Does S' satisfy all identities of S?

There is an obvious connection between the Variety Membership problem and the well-known Finite Basis problem for finite semigroups.

Finite basis problem

Recall that a finite semigroup S is said to be finitely based if there exists a finite set Σ of its identities such that every identity of the semigroup S can be deduced from identities from Σ. The semigroup which is not finitely based is called nonfinitely based.

If a semigroup is finitely based, then the membership problem of the variety it generates has an easy solution of polynomial time complexity.

Our construction

We have constructed the 6-element semigroup GA_2, which is:

- nonfinitely based;
- the complexity of $\textsc{Var-Memb}(GA_2)$ is at most quadratic;
- it is the smallest such an example as 6 is the minimum number of elements in a nonfinitely based semigroup.

Indeed, we add to the semigroup known as $A_2 = \langle a, b \mid a^2 = a, b^2 = 0, \ aba = a, \ bab = b \rangle$ a new element g and define the multiplication by putting $xg = gx = g$ for all $x \in A_2$ and $g^2 = 0$.

Equational complexity

The equational approach leads to the notion of equational complexity β_S of a variety, generated by a finite semigroup S.
$\beta_S(k)$ equals to a minimal n such that for every

$$S' \notin var(S) \text{ such that } |S'| \leq k$$

there is an identity of size smaller then n that holds in S and fails in S'.

This complexity measure was introduced by G. McNulty and Z. Székely.

Equational complexity

In some sense, the complexity of Variety Membership problem VAR-MEMB(S) can have two sources:

- either we need to check too much and too long identities in the input semigroup S' due to the equational complexity $\beta_S(|S'|)$
- or the identities themselves need a lot of calculations for their checking in S' (in the worth case it needs $|S'|^{\beta_S(|S'|)|}$).

So, another problem, which we are interested in, is an Identity Checking in finite semigroups.

Identity Checking Problem

Given finite semigroup S, the Identity Checking problem in S, CHECK-ID(S), is a combinatorial decision problem with:

INSTANCE: semigroup identity $u = v$.

QUESTION: Does S satisfy the identity $u = v$?

The straightforward algorithm enumerating all identity assignments requires exponential time $|S|^{|u|+|v|}$ of the input data.

For any finite semigroup S the problem CHECK-ID(S) belongs to the complexity class co-NP.

Complexity of CHECK-ID

The semigroup examples, constructed by McKenzie and Jackson and mentioned above have co-NP-complete Idenity Checking problem.

CHECK-ID(GA_2) is polynomially decidable. The reason is $var(GA_2) = var(A_2 \times \mathbb{C}_2)$.

Complexity of CHECK-ID

In literature one normally encounters two other examples of 6-element nonfinitely based semigroups – the monoids A_2^1 and B_2^1. However, the complexity of $\textsc{Var-Memb}(A_2^1)$ and $\textsc{Var-Memb}(B_2^1)$ is still unknown.

While the complexity of Checking Identities for these monoids is already known: $\text{CHECK-ID}(A_2^1) \in \text{P}$ (Szabo, Seif) and CHECK-ID(B_2^1) is co-NP-complete (independently Klima and Seif).

Conclusion

	Finite basis	CHECK-ID $\in$ P	VAR-MEMB $\in$ P
Jackson, McKenzie	$-$	$-$	$-$
$\mathbb{A}_5$	$+$	$-$	$+$
GA_2	$-$	$+$	$+$
A_2^1	$-$	$+$	?
B_2^1	$-$	$-$	?
?	$-$	$+$	$-$
?	$-$	$-$	$+$

Thank you for your attention!

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Identity basis for GA_2

$$x^2 = x^4,$$

$$xyx = (xy)^3 x,$$

$$xyxzx = xzxyx,$$

$$\forall n = 1, 2, \ldots \qquad (x_1^2 x_2^2 \cdots x_n^2)^2 = (x_1^2 x_2^2 \cdots x_n^2)^3.$$