Constraint Satisfaction and Width

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Constraint Satisfaction Problems

Let $\mathbf{B} = (B; P_1^{\mathbf{B}}, \ldots, P_m^{\mathbf{B}})$ be a relational structure

Def: $\mathrm{CSP}(\mathbf{B})$ is the following computational problem:

- Input: A structure $\mathbf{A} = (A; P_1^{\mathbf{A}}, \ldots, P_m^{\mathbf{A}})$
- Output: Is there an homomorphism from $\mathbf{A}$ to $\mathbf{B}$?

An *homomorphism* is any mapping $h : A \to B$ such that for every $i \leq m$ and every $(a_1, \ldots, a_n) \in A^n$

$$(a_1, \ldots, a_n) \in P_i^{\mathbf{A}} \Rightarrow (h(a_1), \ldots, h(a_n)) \in P_i^{\mathbf{B}}$$

If such h exists, we write $\mathbf{A} \to \mathbf{B}$

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- If k-CLIQUE is a complete graph with k nodes then CSP(k-CLIQUE) is the GRAPH k-COLORING problem
- If $\mathbf{B}_{\text{3-SAT}}$ is $(\{0,1\}; R_0, R_1, R_2, R_3)$ where

$$R_0 = \{0,1\}^3 - \{(0,0,0)\}$$
$$R_1 = \{0,1\}^3 - \{(1,0,0)\}$$
$$R_2 = \{0,1\}^3 - \{(1,1,0)\}$$
$$R_3 = \{0,1\}^3 - \{(1,1,1)\}$$

then CSP($\mathbf{B}_{\text{3-SAT}}$) is 3-SAT.

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Recall that 3-SAT is the computational problem

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3-SAT example expanded

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It is easy to define a bijection σ between 3-CNF's and structures $\mathbf{A}$ that preserves satistiability and unsatisfiability

Indeed, let $\sigma(\varphi) = (V, T_0, T_1, T_2, T_3)$ where

- V is the set of variables of φ
- T_0 contains (x, y, z) if $x \vee y \vee z$ is a clause of φ
- T_1 contains (x, y, z) if $\neg x \vee y \vee z$ is a clause of φ
- T_2 contains (x, y, z) if $\neg x \vee \neg y \vee z$ is a clause of φ
- T_3 contains (x, y, z) if $\neg x \vee \neg y \vee \neg z$ is a clause of φ

Complexity

For every $\mathbf{B}$, $\mathrm{CSP}(\mathbf{B})$ is in NP.

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Research Project: Identify, for each $\mathbf{B}$, the computational complexity (in P, NP-complete, in NL, in L) of $\mathrm{CSP}(\mathbf{B})$

- A long list of partial results but still open

Two main algorithmic principles to identify tractable(=solvable in polynomial time) cases of $\mathrm{CSP}(\mathbf{B})$

- Few subalgebras property [Berman, Idziak, Markovic, McKenzie, Valeriote, Willard] (P. Markovic talk)
- Bounded Width (this talk)

Two main algorithmic principles to identify tractable(=solvable in polynomial time) cases of $\mathrm{CSP}(\mathbf{B})$

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- Bounded Width (this talk)

Challenge: Investigate how these two principles can be sistematically combined.

Bounded width

The notion of bounded width admits several alternative characterizations:

- in terms of solvability by the k-consistency test
- in terms of obstruction sets
- in terms of definability in certain logics

First view: the k-consistency test

Given $k \geq 1$, $\mathbf{A}$ and $\mathbf{B}$

Let H be the set of all partial homomorphisms f with $\mathrm{dom}(h) \leq k$

Repeat (1) and (2) until stabilizes

1. Remove from H every f with $\mathrm{dom}(f) < k$ such that for some $a \in A$ there is not $g \in H$ with $f \subseteq g$ and $a \in \mathrm{dom}(f)$
2. Remove from H every f such that $g \subseteq f$ for some $g \notin H$

If $H = \emptyset$ then REJECT, otherwise ACCEPT

- If k is fixed the k-consistency test runs in polynomial time

Question 1: Given an structure $\mathbf{B}$ and some $k > 1$, does the k-consitency test solve $\mathrm{CSP}(\mathbf{B})$? that is, does every instance that passes the k-consistency test have a solution?

Second view: Obstruction sets

[Nešetřil, Pultr 78]

Obvious fact: if $\mathbf{O} \to \mathbf{A}$ and $\mathbf{O} \not\to \mathbf{B}$ then $\mathbf{A} \not\to \mathbf{B}$

Def: An obstruction set for a structure $\mathbf{B}$ is a class $\mathcal{O}_{\mathbf{B}}$ of structures such that, for all $\mathbf{A}$,

$$\mathbf{A} \to \mathbf{B} \Leftrightarrow \mathbf{O} \not\to \mathbf{A} \text{ for all } \mathbf{O} \in \mathcal{O}_{\mathbf{B}}$$

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- Every structure $\mathbf{B}$ has a trivial obstruction set containing all $\mathbf{O}$ such that $\mathbf{O} \not\rightarrow \mathbf{B}$
- We are interested in those $\mathbf{B}$ for which is possible to obtain “simple” obstruction sets.

Examples

- If B is a transitive tournament $\overrightarrow{\mathbf{T}_k}$ on k vertices then one can choose $\mathcal{O}_{\mathbf{B}} = \{\overrightarrow{\mathbf{P}_{k+1}}\}$ where $\overrightarrow{\mathbf{P}_{k+1}}$ is a directed path on $k+1$ vertices.

Examples

Algorithm for CSP($\overrightarrow{\mathbf{T}_k}$)

1. input: directed graph $\overrightarrow{\mathbf{A}} = (V, E)$
2. $C_1 := V$
3. $i := 1$
4. while $i \leq k$ do
   4.1 $C_{i+1} := \{v \in V \mid u \in C_i, (u, v) \in E\}$
   4.2 $i := i + 1$
5. if $C_{k+1} = \emptyset$ then ACCEPT, otherwise REJECT

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- If B is 2-CLIQUE then $\mathcal{O}_{\mathrm{B}}$ can be chosen to consist of all odd cycles.

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- G can be obtained from a k-tree G' by choosing a k-clique of G' and adding a new element adjacent to them.

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Question 2: Given a structure $\mathbf{B}$ and some $k > 1$, has $\mathbf{B}$ an obstruction set consisting of structures with tw $\leq k$?

Third view: Logic

Let $\mathbf{O} = (O; P_1^{\mathbf{O}}, \ldots, P_m^{\mathbf{O}})$ be an structure with signature $\{P_1, \ldots, P_m\}$.

Def: $F_{\mathbf{O}}$ is the primitive positive (only existential quantification and conjuntions) sentence in prefix normal form with

- variables of $F_{\mathbf{O}}$ are elements in the universe of $\mathbf{O}$
- there is an atomic predicate $P_i(v_1, \ldots, v_k)$ for every tuple $(v_1, \ldots, v_k) \in P_i^{\mathbf{O}}$

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Example: $F_{\overrightarrow{\mathbf{P}_{k+1}}}$ is the formula

$$\exists v_1, \ldots, v_{k+1} E(v_1, v_2) \wedge \cdots \wedge E(v_k, v_{k+1})$$

Fact: [Chandra, Merlin 77]
For every $\mathbf{A}$, $\mathbf{O}$

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If B has a finite obstruction set $\{\mathbf{O}_1, \ldots, \mathbf{O}_m\}$ then $\neg\,\mathrm{CSP}(\mathbf{B})$ is definable in existential positive FO

$$\mathbf{A} \in \neg\,\mathrm{CSP}(\mathbf{B}) \Leftrightarrow \mathbf{A} \models F_{\mathbf{O}_1} \vee \cdots \vee F_{\mathbf{O}_m}$$

We shall write existential positive formulas in form of rules

Example: The formula

$$\exists x, y, z \quad E(x,y) \wedge E(y,z) \wedge (z,x)$$
$$\vee$$
$$\exists x, y \quad E(x,y) \wedge E(y,x)$$

can be rewriten as

$$\text{Goal} \quad :- \quad E(x,y), E(y,z), E(z,x)$$
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Intuition: “Goal” is fired when the right side of a rule is satisfied.

Datalog Programs

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The intensional database predicates (IDB) might occur both in the *head* and *body* of a rule

Question 3: Given a structure $\mathbf{B}$ and $k > 1$, is $\neg\,\mathrm{CSP}(\mathbf{B})$ definable by a Datalog Program with at most k different variables?

[Hell, Nešetřil, Zhu 96][Feder,Vardi 98][Kolaitis,Vardi 00]

Theorem: Let $\mathbf{B}$ be a structure and $k \geq 1$. Tfae:

- k-consistency solves $\mathrm{CSP}(\mathbf{B})$
- $\mathbf{B}$ has an obstuction set consisting of structures of trewidth $\leq k-1$
- $\neg\,\mathrm{CSP}(\mathbf{B})$ is definable by a datalog program with k different variables

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Def:

$\mathbf{B}$ has width k if it satisfies any of the previous conditions

$\mathbf{B}$ has bounded width if has width k for some k

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Algebraic approach

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Fact: Many properties of $\mathrm{CSP}(\mathbf{B})$ depend only on $\mathrm{Alg}_{\mathbf{B}}$

- Solvability in poly time [Jeavons, Cohen, Gyssens 98]
- Bounded width and many others [Larose, Tesson 07]

Sufficient conditions: $\mathbf{B}$ has bounded width if $\mathrm{Alg}_{\mathbf{B}}$

- has a semilattice [Jeavons, Cohen, Gyssens 97]
- has a nu [Feder, Vardi 98]
- has 2-semilattice [Bulatov 06]
- is in CD(3) [Kiss,Valeriote 07]
- $\ldots$

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- $\mathrm{Alg}_{\mathbf{B}}$ has weak nufs of almost all arities [Maróti, McKenzie 07]

An idempotent operation f of arity $n \geq 2$ is a weak nuf it it satisfies the identity

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It is conjectured [Larose,Zadori 07] that the condition is also sufficient

Width from an algebraic perspective

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Solution: Work with a simplified version of width

Width of an algebra

Let $\mathcal{A}$ be an algebra, let $n \geq k > 2$, let H be a subuniverse of $\mathcal{A}^n$, and let $\{H_I : I \subseteq \{1, \ldots, n\}, |I| = k\}$ the set of all its k-ary projections

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Def: $\mathcal{A}$ has width k if for every k-relation system there is a tuple $t \in A^n$ such that $t_I \in H_I$ for every I.

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- all algebras known to be bounded have with k for almost all k

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- Has every algebra in CD(4) have width k for some $k > 2$?
- Is it true that if $\mathbf{B}$ has width k then it has width $k + 1$?
- Is there an algebra that has width k for some $k > 3$ but not width 3?

Interesting cases of obstruction sets

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Obstructions of bounded pathwidth

Theorem: (D. 05)
The following conditions are equivalent:

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A datalog program is *linear* if it has at most one IDB in the body of each rule.

Example: The following program is linear

$$\begin{aligned} \text{oddpath}(X,Y) &:- E(X,Y) \\ \text{oddpath}(X,Y) &:- \text{oddpath}(X,Z), E(Z,T), E(T,Y) \\ \text{non2colorable} &:- \text{oddpath}(X,X) \end{aligned}$$

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Theorem: If $\mathbf{B}$ is a core with an obstruction of bounded pathwidth then $\mathrm{var}(\mathrm{Alg}_{\mathbf{B}})$ omits types 1,2, and 5 [Larose, Tesson 07]

Fact: If $\mathbf{B}$ has an obstruction set of bounded pathwidth then $\neg\,\mathrm{CSP}(\mathbf{B})$ is solvable with non-deterministic logarithmic space (NL)

Indeed, all CSPs known to be in NL have bounded pathwidth duality

Theorem: If $\mathbf{B}$ is a core with an obstruction of bounded pathwidth then $\mathrm{var}(\mathrm{Alg}_{\mathbf{B}})$ omits types 1,2, and 5 [Larose, Tesson 07]

Theorem: If $\mathbf{B}$ is invariant under a majority then it has an obstruction set of bounded pathwidth [D., Krokhin 07]

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- Does every $\mathbf{B}$ with $\mathrm{Alg}_{\mathbf{B}}$ in CD(3) have an obstruction set of bounded pathwidth?
- Does it exists any $\mathbf{B}$ without an obstruction set of bounded patwidth such that $\neg\,\mathrm{CSP}(\mathbf{B})$ is in NL.

Trees

Theorem [Feder, Vardi 98]. Let $\mathbf{B}$ be a structure. Tfae:

- $\mathbf{B}$ has an obstruction set consisting of trees
- $\neg\,\mathrm{CSP}(\mathbf{B})$ is definable by a Datalog program with monadic IDBs and with at most one EDB per rule.
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Theorem [D., Krokhin] Let $\mathbb{B}$ be a structure. Tfae:

- $\mathbb{B}$ has an obstruction set consisting of caterpillars.
- $\neg\,\mathrm{CSP}(\mathbb{B})$ is definable by a Dat. program with monadic IDBs and with at most one EDB and one IDB per rule.
- $\mathbb{B}$ is a retract of a structure invariant under a lattice.

A caterpillar is a tree in which every node is adjacent to at most 2 non-leaves

Why is it possible in the case of trees and caterpillars to find an exact algebraic characterization?

Answer: there is a methodology, due to Feder and Vardi.

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1. Define a notion of consistency that captures the type of obstruction considered

First stage

The restriction of the k-consistency test that corresponds to trees is the arc-consistency test.

Arc consistency test.

Input $\mathbf{A} = (A; P_1^{\mathbf{A}}, \ldots, P_l^{\mathbf{A}}, \mathbf{B} = (B; P_1^{\mathbf{B}}, \ldots, P_l^{\mathbf{B}})$:

Let H be the mapping $A \to 2^B$ such that $H(a) = B$ for all a.

1. For every P_i, every tuple $(a_1, \ldots, a_r) \in P_i^{\mathbf{A}}$, and every $1 \leq j \leq r$ remove from $H(a_j)$ all those values not in $\mathrm{pr}_j\, R^{\mathbf{B}} \cap H(a_1) \times \cdots \times H(a_r)$

Iterate (1) until stabilizes

If $H(a) = \emptyset$ for some $a \in \mathbf{A}$ then REJECT otherwise accept

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1. Define a notion of consistency that captures the type of obstruction considered
2. Find out the *most difficult example* (structure) for the corresponding consistency test

Second stage

The most difficult example for the arc-consistency test is structure $U(\mathbf{B})$.

Def: $U(\mathbf{B})$ is the structure whose nodes are nonempty sets of B and such that for every P_i, $P_i^{U(\mathbf{B})}$ contains $(\mathrm{pr}_1 R, \ldots, \mathrm{pr}_r R)$ for every subrelation R of $P_i^{\mathbf{B}}$.

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If f and g are homomorphisms from $\mathbf{A}$ to $U(\mathbf{B})$ then so is $h(a) = f(a) \cup g(a)$

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H is precisely the maximal homomorphism, if exists.

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Hence,

$$\text{Arc-consistency solves } \mathrm{CSP}(\mathbf{B}) \Leftrightarrow U(\mathbf{B}) \rightarrow \mathbf{B}$$

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1. Define a notion of consistency that captures the type of obstruction considered
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3. Algebraic characterization follows from the analysis of the structure.

Third stage

Finally, tfae:

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$(\Leftarrow)$ If $\mathbf{B}$ is invariant under a semilattice $\vee$ then $h(\{a_1, \ldots, a_n\}) = \vee\{a_1, \ldots, a_n\}$ is an homomorphism from $U(\mathbf{B})$ to $\mathbf{B}$.

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Furthermore if B satisfies any of the previous conditions then tfae [Loten, Tardif]:

- B has a finite obstruction set consiting of caterpillars
- B is invariant under a majority

THANKS FOR YOUR ATTENTION!!!!

For more details on this see:

- Slides of L.Zadori's talk at Vanderbilt
- Upcoming survey by A.Bulatov, A. Krokhin, and B. Larose