Contents

Equaclosure operators on finite lattices

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We investigate approaches to the solution of Birkhoff-Mal'cev problem about the description of (finite) lattices of quasivarieties. All known finite lattices of quasivarieties are lower bounded. Also, every lattice of quasivarieties admits a so-called equaclosure operator. We work toward description of lower bounded lattices that admit an equaclosure operator. In particular, we build an algorithm that works for any finite lattice and either decides that the lattice does not admit an equaclosure operator, or generates the minimal equaclosure operator on it.

Joint work with J. B. Nation (University of Hawaii).

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Abelian groups determined by the subgroup lattice of direct powers

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We show that the class of abelian groups determined by the subgroup lattices of their direct squares is exactly the class of the abelian groups which share the square root property. As application we answer in the negative a (semi)conjecture of Palfy and solve a more general problem.

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Axiomatizability by sentences of the form $\forall\exists! \wedge p = q$

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Given an equational class $\mathcal{V}$, several important subclasses of $\mathcal{V}$ can be defined by sentences of the form $\forall\exists! \wedge p = q$. For example, if $\mathcal{V}$ is the class of all semigroups with unit, then the subclass of all groups can be defined in this way, and if $\mathcal{V}$ is the class of all bounded distributive lattices, then the subclass of all Boolean lattices is also axiomatizable in this way. We consider the following general problem:

Problem: Given a variety $\mathcal{V}$ characterize the subclasses of $\mathcal{V}$ which can be axiomatized by a set of sentences of the form $\forall\exists! \wedge p = q$.

We will present a solution to this problem for certain varieties of distributive lattice expansions. Joint work with Diego Vaggione.

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Semilattices of finite Moore families

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Moore families (also called *closure systems*) are set representations for lattices. For instance, the Moore families called *convex geometries* represent lower locally distributive lattices (they are also in duality with the path-independent choice functions of the consumer theory in microeconomics). We present a review of a number of works on some sets of Moore families defined on a finite set S which, ordered by set inclusion, are semilattices or lattices. In particular, we study the lattice $\mathcal{M}_P$ (respectively, the semilattice $\mathcal{G}_P$) of all Moore families (respectively, convex geometries) having the same poset P of join-irreducible elements. For instance, we determine how one goes from a family $\mathcal{F}$ in these lattices (or semilattices) to another one covered by $\mathcal{F}$ and also the changes induced in the irreducible elements of $\mathcal{F}$. In the case of convex geometries, this allows us to get an algorithm computing all the elements of $\mathcal{G}_P$. At last, we characterize the posets P for which $|\mathcal{M}_P|$ or $|\mathcal{G}_P|$ is less than or equal to 2.

Common work with Gabriela Hauser Bordalo Universidade de Lisboa, Portugal).

References

[1] K.V. Adaricheva, *Characterization of finite lattices of sublattices* (in Russian), Algebra i logica **30**, 1991, 385–404.

[2] G. Bordalo and B. Monjardet, *The lattice of strict completions of a finite lattice*, Algebra Universalis **47**, 2002, 183–200.

[3] G. Bordalo and B. Monjardet, *Finite orders and their minimal strict completions lattices*, Discussiones Mathematicae, General Algebra and Applications **23**, 2003, 85–100.

[4] N. Caspard and B. Monjardet, *The lattice of closure systems, closure operators and implicational systems on a finite set: a survey*, Discrete Applied Mathematics **127(2)**, 2003, 241–269.

[5] N. Caspard and B. Monjardet, *The lattice of convex geometries*, in M. Nadif, A. Napoli, E. SanJuan, A. Sigayret, Fourth International Conference "Journés de l'informatique Messine", Knowledge Discovery and Discrete Mathematics, Rocquencourt, INRIA, 2003, 105–113.

[6] N. Caspard and B. Monjardet, *Some lattices of closure systems*, Discrete Mathematics and Theoretical Computer Science **6**, 2004, 163–190.

[7] M.R. Johnson and R.A. Dean, *Locally Complete Path Independent Choice functions and Their Lattices*, Mathematical Social Sciences **42(1)**, 2001, 53–87.

[8] B. Monjardet and V. Raderanirina, *The duality between the anti-exchange closure operators and the path independent choice operators on a finite set*, Mathematical Social Sciences **41(2)**, 2001, 131–150.

[9] J.B.Nation and A. Pogel, *The lattice of completions of an ordered set*, Order **14(1)**, 1997, 1–7.

[10] B. Seselja and A. Tepavcevic, *Collection of finite lattices generated by a poset*, Order **17**, 2000, 129–139.

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Orthomodular semilattices

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A semilattice $\mathcal{S} = (\mathcal{S}; \wedge, \mathbf{0})$ with the least element $\mathbf{0}$ is called **orthosemilattice** if the interval $[\mathbf{0}, a]$ is an ortholattice for each $a \in S$. $\mathcal{S}$ is called an **orthomodular semilattice** if $[\mathbf{0}, a]$ is an orthomodular lattice for each $a \in S$. We will present simple identities characterizing varieties of these semilattices. The so-called compatibility condition will be discussed.

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On the lattice of equational classes of operations and its monoidal intervals

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Let A be a finite non-empty set. By a *class* of operations on A we simply mean a subset $\mathcal{I} \subseteq \cup_{n\geq 1} A^{A^n}$. The *composition* of two classes $\mathcal{I}, \mathcal{J} \subseteq \cup_{n\geq 1} A^{A^n}$ of operations on A, denoted $\mathcal{I}\mathcal{J}$, is defined as the set

$$\mathcal{I}\mathcal{J} = \{f(g_1, \ldots, g_n) \mid n, m \geq 1, f \ n\text{-ary in } \mathcal{I},\ g_1, \ldots, g_n \ m\text{-ary in } \mathcal{J}\}.$$

A *functional equation* (for operations on A) is a formal expression

$$h_1(\mathbf{f}(g_1(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, \mathbf{f}(g_m(\mathbf{v}_1, \ldots, \mathbf{v}_p))) = h_2(\mathbf{f}({g'}_1(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, \mathbf{f}({g'}_t(\mathbf{v}_1, \ldots, \mathbf{v}_p)))$$

where $m, t, p \geq 1$, $h_1 : A^m \to A$, $h_2 : A^t \to A$, each g_i and ${g'}_j$ is a map $A^p \to A$, the $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are p distinct vector variable symbols, and $\mathbf{f}$ is a function symbol. An n-ary operation f on A is said to *satisfy* the above equation if, for all $v_1, \ldots, v_p \in A^n$, we have

$$h_1(f(g_1(v_1, \ldots, v_p)), \ldots, f(g_m(v_1, \ldots, v_p))) = h_2(f({g'}_1(v_1, \ldots, v_p)), \ldots, f({g'}_t(v_1, \ldots, v_p)))$$

The classes of operations definable by functional equations (*equational classes*) are known to be exactly those classes $\mathcal{I}$ satisfying $\mathcal{I}\mathcal{O}_A = \mathcal{I}$, where $\mathcal{O}_A$ denotes the class of all projections on A (for $|A| = 2$ see [4],[5], and for $|A| \geq 2$ see [1]). In particular, clones of operations, i.e. classes containing all projections and idempotent under class composition, are equational classes.

The set of all equational classes on A constitute a complete lattice under union and intersection. Moreover, it is partially ordered monoid under class composition, with identity $\mathcal{O}_A$, and whose non-trivial idempotents are exactly the clones on A. But the classification of operations into equational classes is much finer than the classification into clones: for $|A| = 2$, there are uncountably many equational classes on A, but only countably many of them are clones. Also, the set of clones does not constitute a monoid since it is not closed under class composition (see [2]).

The aim of this presentation is to investigate the lattice of equational classes of operations on a finite set A. For $|A| = 2$, we classify all monoidal intervals $[\mathcal{C}_1, \mathcal{C}_2]$, for clones $\mathcal{C}_1$ and $\mathcal{C}_2$, in terms of their size: we give complete descriptions of the countable intervals, and provide families of uncountably many equational classes in the remaining intervals. In particular, from this classification it will follow that an interval $[\mathcal{C}_1, \mathcal{C}_2]$ contains uncountably many equational classes if and only if $\mathcal{C}_2 \setminus \mathcal{C}_1$ contains a "non-associative" Boolean function.

References

[1] M. Couceiro, S. Foldes, *Constraints, Functional Equations, Definability of Function Classes, and Functions of Boolean Variables*, Rutcor Research Report 36-2004, Rutgers University, http://rutcor.rutgers.edu/ rrr/.

[2] M. Couceiro, S. Foldes, E. Lehtonen, *Composition of Post Classes and Normal Forms of Boolean Functions*, Rutcor Research Report 05-2005, Rutgers University, http://rutcor.rutgers.edu/ rrr/.

[3] O. Ekin, S. Foldes, P. L. Hammer, L. Hellerstein, *Equational Characterizations of Boolean Functions Classes*, Discrete Mathematics, **211** (2000) 27–51.

[4] N. Pippenger, *Galois Theory for Minors of Finite Functions*, Discrete Mathematics, **254** (2002) 405–419.

[5] E. L. Post, *The Two-Valued Iterative Systems of Mathematical Logic*, Annals of Mathematical Studies **5** (1941) 1–122.

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Rank is not rank!

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In 1997, Ross Willard [2] introduced a powerful but extremely technical condition related to strong dualisability in the theory of natural dualities. He gave a definition by transfinite induction of the **rank** of a finite algebra. The Oxford English Dictionary (OED) gives 12 different interpretations for the noun 'rank', including the one intended by Willard:

position in a numerically ordered series; the number specifying the position.

When I first saw the definition of rank, what immediately came to mind was one of the 15 OED definitions of the adjective 'rank', namely

having an offensively strong smell; rancid.

The aim of this talk is to show that I was completely wrong in my original assessment of the concept of rank. I will show that there is a very natural way to introduce the concept. Along the way, I shall give a characterization of those dualisable algebras that have rank 0, and show how rank 1 is related to both the injectivity of the algebra and to the congruence distributivity of the variety it generates.

The ranks of many finite algebras have been calculated. For example: the three-element Kleene algebra has rank 0, as does the two-element bounded distributive lattice (though its unbounded cousin has rank 1); for each prime p, the ring (with identity) of integers modulo p^2 has rank 1; every finite unar, and more generally every finite linear unary algebra, has rank at most 2; entropic graph algebras and entropic flat graph algebras have rank at most 2; every dualisable algebra that is not strongly dualisable has rank infinity.

There is still much to learn about the notion of rank—between rank 2 and rank infinity, no examples are presently known.

The results presented in this talk are part of the appendix on strong dualisability in a new text by Pitkethly and Davey [1].

References

[1] J. G. Pitkethly and B. A. Davey, *Dualisability: Unary Algebras and Beyond*, Springer, 2005.

[2] R. Willard, *New tools for proving dualizability*, Dualities, Interpretability and Ordered Structures, (Lisbon, 1997), (J. Vaz de Carvalho and I. Ferreirim, eds), Centro de Álgebra da Universidade de Lisboa, 1999, pp. 69–74.

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T-clones and T-hyperidentities

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The aim of this paper is to describe in which way varieties of algebras of type τ can be classified by using the form of the terms which build the (defining) identities of the variety. There are several possibilities to do so. In [1], [5], [4] the authors considered normal identities, i.e. identities which have the form $x \approx x$ or $s \approx t$ where s and t contain at least one operation symbol. This was generalized in [Den-W;04] to k-normal identities and in [2] to P-compatible identities. More general, we select a subset T of $W_\tau(X)$ and consider identities from $T \times T$. Since every variety can be described by one heterogenous algebra, its clone, we are also interested in the corresponding clone-like structure. Since identities of the clone of a variety correspond to M-hyperidentities for certain monoids

M of hypersubstitutions, we will also investigate these monoids and the corresponding M-hyperidentities.

References

[1] I. Chajda, *Normally presented varieties*, Algebra Universalis, **34** (1995), 327–335.

[2] I. Chajda, K. Denecke, S. L. Wismath, *A Characterization of P-compatible Varieties*, preprint 2004.

[3] K. Denecke, S. L. Wismath, *A characterization of k-normal varieties*, Algebra Universalis, **51** (2004), 395–409.

[4] E. Graczyńska, *On normal and regular identities and hyperidentities*, in: Universal and Applied Algebra, Proccedings of the Fifth Universal Algebra Symposium, Turawa (Poland), 1988, World Scientific, 1989, 107–135.

[5] I. I. Melnik, *Nilpotent shifts of varieties* ,(In Russian), Mat. Zametki, Vol. **14**, No. 5 (1973), english translation in: Math. Notes **14** (1973), 962-966.

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On discrete convexity and ordered sets

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Connections between order and convexity will be discussed.

On any ordered set there are several closure operators which exhibit different aspects of the behaviour of the convex hull operator in Euclidean space, such as the anti-exchange property or the separation of points by half-spaces. The latter is related to classical and more recent results, and to some open problems, concerning the representation of partial orders by linear orders.

Monotonicity properties of functions between ordered sets, of real-valued functions on the discrete hypercube in particular, can also be advantageously described from the perspective of convexity. These properties correspond to classes of functions that are themselves convex sets in function space, and closed under some additional conditions. Certain lattices of function properties can be characterized by a small number of closure criteria.

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Computing the tame congruence theory type set of an algebra

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We discuss the algorithm used in our algebra program to compute the type set of an algebra. This is a modification of the algorithm given in [1]. The type set of an algebra $\mathbf{A}$ is the same as the type set of β/β_*, β join irreducible in $\mathbf{Con\ A}$. We find a subtrace for each such β and then find its type.

We give bounds on certain subalgebras of $\mathbf{A}^4$, which show that both our algorithm and the one in [1] are faster than previously thought. We discuss a result distinguishing type **5** from type **4** using a greatest lower bound property of an associated structure.

The talk will be accessible with little or no background in Tame Congruence Theory.

References

[1] Berman, E. Kiss, P. Prőhle, and Á. Szendrei, *The type set of a finitely generated variety*, Discrete Math. **112** (1993), 1–20.

[2] D. Hobby and R. McKenzie, *The structure of finite algebras (tame congruence theory)*, Contemporary Mathematics, American Mathematical Society, Providence, RI, 1988.

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Relations between lattice-properties

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It is well-known that for lattices both associativities are equivalent to the transitivity of the corresponding partially ordered set. It is also well-known that both distributivities are equivalent to the medial identity. We shall prove that "in general" this is not the case. Some related topics will be investigated. As a frame we shall omit all the lattice identities except the eight(!) absorption laws. This is enough to represent the structure (which will be called weak lattice) as a relational system.

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Fluid varieties

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We invent the notion of a *derived variety* as a tool for exploring the lattice of subvarieties of a given variety. A modification of the notion of a *fluid variety* invented in [10] as a counterpart of the notion for a *solid variety* (see [6]) is given. A fluid variety V has no proper derived variety as a subvariety. We examine some properties of derived and fluid varieties in the lattice of all varieties of a given type τ. Examples of such varieties of bands are presented.

Joint work with D. Schweigert (Technische Universität Kaiserslautern).

References

[1] K. Denecke, S. L.Wismath, *Hyperidentities and Clones*, Gordon & Breach, 2000.

[2] T. Evans, *The lattice of semigroups varieties*, Semigroup Forum **2** (1971), 1–43.

[3] Ch. F. Fennemore, *All varieties of bands*, Ph.D. dissertation, Pensylvania State University, 1969.

[4] Ch. F. Fennemore, *All varieties of bands I*, Mathematische Nachrichten, **48**, 1971, 237–252.

[5] J. A. Gerhard, *The lattice of equational classes of idempotent semigroups*, J. of Algebra, **15** (1970), 195–224.

[6] E. Graczyńska and D. Schweigert, *Hyperidentities of a given type*, Algebra Universalis **27** (1990), 305–318.

[7] G. Grätzer, *Universal Algebra.* 2^{nd} ed., Springer, New York 1979.

[8] J. Płonka, *On hyperidentities of some varieties*,in: Genaral Algebra and Discrete Mathematics, eds. K. Denecke, O. Lüders, Heldermann-Verlag-Berlin, 1995, pp. 199–213.

[9] D. Schweigert, *Hyperidentities*, in: I. G. Rosenberg and G. Sabidussi, Algebras and Orders, 1993 Kluwer Academic Publishers, 405–506. ISBN 0-7923-2143-X.

[10] D. Schweigert, *On derived varieties*, Discussiones Mathematicae Algebra and Stochastic Methods 18 (1998), 17–26.

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Weakly standard BCC-algebras

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The notion of a BCK-algebra was introduced in 60's by Imai and Iséki as an algebraic formulation of Meredith's BCK-implicational calculus. Left-distributive BCK-algebras, known as Hilbert algebras, form an algebraic counterpart of the logical connective implication in intuitionistic logic. When solving the problem whether the class of all BCK-algebras forms a variety, Komori introduced the class of BCC- algebras and proved that this class is not a variety. The axioms of BCC-algebras allow us to define a natural order relation on a base set. It is well known that there is no restriction to the corresponding posets in that sense that one can define on every poset a structure of a BCC-algebra. This holds even for Hilbert algebras and the corresponding structures are called order-algebras. Order algebras satisfy a very strong property that every subset containing the distinguished element 1 (considerd as a logical value "true") form a subalgebra. A natural problem to describe all BCC-algebras in which every 3-element subset containing 1 is a subalgebra was solved by the author in 2002, the resulting algebras are here called standard BCC-algebras. The aim of my talk is to present a new construction of BCC-algebras from posets requiring a weaker condition on its subalgebras. Resulting structures are called weakly-standard BCC-algebras.

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Congruence preserving functions on distributive lattices I

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In a joint project with M. Ploščica we initiate a problem of describing the congruence-preserving functions of an algebra in a given (favourite) variety. We describe the congruence-preserving functions of an arbitrary algebra in the variety of bounded distributive lattices $\mathcal{D}_{01}$ and the varieties of distributive lattices with one bound $\mathcal{D}_0$ and $\mathcal{D}_1$. We also present a partial solution in the non-bounded case which will be completed in a continuation of this talk by M. Ploščica.

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The * problem for finite groups

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The complexity of the term-equivalence, polynomial-equivalence and polynomial-satisfiability problems are well-known for rings (Burris – Lawrence and Szabó – Vértesi). These problems are open for groups, there are some results from Burris – Lawrence, Goldmann – Russel and Horváth – Szabó. For semigroups we know almost nothing, the few known results belong to Tesson, Thérien, Volkov, Szabó, Vértesi, Klíma.

A new version of these problems was defined in 2003. The algebra is presented with the operation table of the basic operations. But it can be fruitful to compute other operation tables from the clone of the algebra, in advance (i.e. the commutator-table of the given group) as preprocessing. In this case, our expressions can be shortened, so the complexity of the problem can change.

Let us denote with $\mathrm{Term}-\mathrm{EQ}^*$, $\mathrm{Pol}-\mathrm{EQ}^*$ and $\mathrm{Pol}-\mathrm{SAT}^*$ the former problems when we have the possibility to preprocess finitely many new operations with the basic operations of the algebra. Idziak – Szabó in 2004 proved that if $typ(\mathbf{A}) \subseteq \{2,3\}$, then $\mathrm{Pol}-\mathrm{SAT}^*\,(\mathbf{A})$ is in P for nilpotent Sylow algebras and NP-complete for non-nilpotent algebras. They do not say anything about terms. We investigate these questions for groups:

Theorem (G. Horváth, Cs. Szabó, 2004). *For a finite group G*

- *if G is a nilpotent group then* $\mathrm{Term}-\mathrm{EQ}^*\,(G)$ *is in P;*
- *if G is not nilpotent then* $\mathrm{Term}-\mathrm{EQ}^*\,(G)$ *is coNP-complete.*

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Arithmetical affine complete varieties and inverse monoids

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We call an algebra *minimal* if it has no proper subalgebras and *weakly diagonal* if every subuniverse of its square contains the graph of some of its automorphisms. It is known that every affine complete variety of finite type is generated by a finite weakly diagonal algebra which is unique up to isomorphism and that every finite minimal algebra of arithmetical variety generates affine complete variety.

Let $\mathbf{A}$ be a finite minimal algebra generating arithmetical (hence also affine complete) variety. Then the set of all non-empty subuniverses of $\mathbf{A}^2$ forms an inverse monoid with respect to the relational product operation. The elements of this monoid are called *bicongruences* of $\mathbf{A}$ and the monoid itself is denoted by $\mathbf{BiconA}$.

The algebras $\mathbf{A}$ and $\mathbf{B}$ are said to be *categoricallly equivalent* if there is an equivalence of categories $F : \mathrm{Var}\mathbf{A} \to \mathrm{Var}\mathbf{B}$ such that $F(\mathbf{A}) = \mathbf{B}$. Two varieties are called categorically equivalent if they have categorically equivalent generators.

Theorem 1. Two finite minimal algebras generating arithmetical varieties are categorically equivalent iff their monoids of bicongruences are isomorphic.

We would like to describe the monoids appearing in Theorem 1 but so far we have not been able to solve this problem. However, we were able to handle the weakly diagonal case. Note that an inverse monoid $\mathbf{S}$ is called *factorizable* if for every $s \in S$ there exists a unit g of $\mathbf{S}$ such that $s \leq g$ (here $\leq$ is the natural order relation of $\mathbf{S}$).

Theorem 2. Given a finite inverse monoid $\mathbf{S}$, the following are equivalent:

[1] there exists a finite weakly diagonal algebra $\mathbf{A}$ such that $\mathrm{Var}\mathbf{A}$ is arithmetical and $\mathbf{BiconA} \simeq \mathbf{S}$;

[2] $\mathbf{S}$ is factorizable, has a multiplicative zero, its idempotents form a distributive lattice under the natural order relation and the following condition holds:

(IU) for every two idempotents e_1, e_2 and a unit g of $\mathbf{S}$, if $e_1 e_2 \leq g$ then there exist units g_1, g_2 of $\mathbf{S}$, such that $e_1 \leq g_1$, $e_2 \leq g_2$ and $g_1 g_2 = g$.

This theorem yields a 1-1 correspondence between categorical equivalence classes of arithmetical affine complete varieties of finite type and isomorphism classes of monoids satisfying conditions (2) in Theorem 2.

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Interpolation in modules over simple rings

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We investigate a concept of "polynomial completeness" that was introduced by P. Idziak and K. Słomczyńska. They have called an algebra *polynomially rich* if each function that preserves congruences and the types of prime quotients of the congruence lattice in the sense of Tame Congruence Theory is a polynomial function. On an expanded group, a function preserves types if and only if it preserves certain 4-ary relations. Hence, one can define when a partial function preserves types. We call an expanded group *strictly k-polynomially rich* if each k-ary partial type preserving function is polynomial. In order to give a characterizaton of finite strictly k-polynomially rich expanded groups, it is important to characterize strictly k-polynomially rich modules over simple rings. The contribution of the present talk is such a characterization.

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Abelian relatively modular quasivarieties

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About fifteen years ago, Ralph McKenzie and I partially extended the commutator theory for congruence modular varieties to a similar theory for relatively modular quasivarieties. Our theory lacked a structure theorem for abelian algebras and congruences. In this talk I will present results that fill in this gap in the theory.

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Mal'tsev conditions and centrality

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If an algebra has a majority term, then its congruence lattice is distributive, and no proper congruence can centralize itself. Thus, a Mal'tsev-condition can influence the shape of the congruence lattice, and the behavior of the commutator operation as well. A systematic study of this phenomenon for locally finite varieties has been given by David Hobby and Ralph McKenzie, using tame congruence theory. It turned out later, by the work of K. Kearnes, Á. Szendrei and the author, that parts of this classification can be carried over to the general, nonfinite case via different methods. A new type of centrality, called rectangularity, played an important role in these investigations. The lecture gives an introduction to this topic, and an overview of some of these new results.

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Complexity of checking identities in monoids of transformations

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We study the computational complexity of checking identities and solving equations in a fixed finite monoid. There is a six-element monoid for which these problems are coNP-complete. This monoid can be easily described as a submonoid of the monoid of partial transformations of a two-element set. In the talk we show other natural small monoids of transformations for which the problems are hard.

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On certain sublattices of the lattice of all varieties of semigroups

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Although the lattice of all semigroup varieties has attracted much research attention, very little is now known about its structure, and certain sublattices are known to be be very complicated as it does not satisfy any of the lattice laws. We have made use of a countably infinite family of injective endomorphisms on the lattice of all semigroup varieties, introduced by the author, to describe certain important sublattices of the lattice. Together with other well known results, this information enables us to give a sketch of the lower part of the lattice of all semigroup varieties.

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Non-deterministic hypersubstitutions

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A mapping σ which assigns to every n-ary operation symbol f an n-ary term of a given type τ is said to be a hypersubstitution of type τ. The concept of a hypersubstitution was introduced by Denecke, Lau, Pöschel and Schweigert in the 90th. Every hypersubstitution σ of type τ induces a mapping $\widehat{\sigma}$ on the set of all terms of type τ. Its application to a term t is that term which we get from t by replacing of each operation symbols f_i in t by the term $\sigma(f_i)$. This process will be more clear if we regard a term as a tree. Now we want to consider the case that one has more than one possibility for the replacement of any operation symbol in a given term t by only one application of a mapping to t. This can be realized by the concept of a non-deterministic hypersubstitution. Here the image of any operation symbol f_i is an element of a given set of terms instead of that for f_i determined term $\sigma(f_i)$. That means, the application of a non-deterministic hypersubstitution to a term gives a set of terms. We will introduce the theory of a non-deterministic hypersubstitution as well as of a non-deterministic solid (nd-solid) variety. Further, we give a relationship to the clone theory.

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Hypergraph homomorphisms and compositions of Boolean functions with clique functions

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For a class $\mathcal{C}$ of Boolean functions, we say that a Boolean function f is a $\mathcal{C}$*-subfunction* of a Boolean function g, denoted $f \preceq_{\mathcal{C}} g$, if $f = g(h_1, \ldots, h_n)$, where all the inner functions h_i are members of $\mathcal{C}$. The $\mathcal{C}$-subfunction relation is a preorder on the set Ω of all Boolean functions if and only if $\mathcal{C}$ is a clone, and it is natural to ask whether there is an infinite descending chain of $\mathcal{C}$-subfunctions and what is the size of the largest antichain of $\mathcal{C}$-incomparable functions.

In this presentation, we focus on the clones U_k of all 1-*separating functions* (or *clique functions*) of rank k (for $k = 2, \ldots, \infty$). We characterize the U_k-subfunction relation in terms of homomorphisms between hypergraphs.

For a function f, we define the *rank-*k *disjointness hypergraph of* f, denoted $G(k, f)$, as follows: the vertices of $G(k, f)$ are the true points of f and $S \in E(G(k, f))$ if and only if $2 \leq |S| \leq k$ and $\bigwedge S = \mathbf{0}$. For $k = 2$, this is an ordinary graph—in fact it is the complement of the intersection graph of the true points of f—and we call it the *disjointness graph of* f. It is well-known that every graph is an intersection graph; hence every graph is the rank-2 disjointness graph of some function.

Theorem. *Let f and g be 0-preserving functions. Then $f \preceq_{U_k} g$ if and only if $G(k, f)$ is homomorphic to $G(k, g)$.*

For each k, there is an infinite sequence of hypergraphs $G_1^k, G_2^k, \ldots$ such that G_i^k is the rank-k disjointness hypergraph of some function and, for all i, G_{i+1}^k is homomorphic to G_i^k but G_i^k is not homomorphic to G_{i+1}^k. By Theorem , there is an infinite descending chain of U_k-subfunctions.

There exists also an infinite family of graphs that are pairwise incomparable in the sense that they are not homomorphic to each other. Theorem implies that there exists an infinite antichain of U_2-incomparable functions and hence an infinite antichain of U_k-incomparable functions for any $k \geq 2$.

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Centralizers of monoids containing the symmetric group

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For a monoid M of k-valued unary functions, the centralizer M^* of M is the set of k-valued multi-variable functions which commute with every function in M. In this talk, we determine centralizers for all monoids which contain the symmetric group. For most of such monoids the centralizer turns out to be the least clone. However, there is an exceptional case for $k = 4$ where the centralizer of the monoid called M_2 is not the least clone. Furthermore, by generalizing M_2, we define the monoid M_n of linear unary functions on $\mathbf{2}^n$ and characterize its centralizer.

Joint work with Ivo G. Rosenberg (Université de Montréal, Canada).

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Geometric lattices and the connectivity of matroids of arbitrary cardinality

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Using geometric lattice theory, the concept of connectivity of a finite matroid is successfully generalized to that of a matroid of arbitrary cardinality. Afterwards, three criteria are presented for the connectivity of a matroid of arbitrary cardinality by the help of geometric lattice theory.

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Clones containing the polynomial functions on groups of order pq

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P. M. Idziak has conjectured that each cyclic group of squarefree order has only finitely many polynomially inequivalent expansions. Furthermore, he conjectured that the clone of polynomial functions on every such expansion $\mathbf{V}$ is uniquely determined by the congruences of $\mathbf{V}$ and their commutators.

We show that both assertions are true for each cyclic group whose order is the product of 2 distinct primes. In this case, the polynomial functions on an expansion $\mathbf{V}$ can be obtained by techniques that are already described in the literature except if $\mathbf{V}$ is nilpotent and non-abelian. We solve this remaining case by using module theory.

This is joint work with E. Aichinger.

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Finite basis problems for quasivarieties, and the weak extension property

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We have announced that every finitely generated quasivariety of finite signature whose relative congruence lattices are meet semi-distributive is finitely axiomatizable. The weak extension property (WEP) for quasivarieties plays a role in the proof of this result. A quasivariety $\mathcal{K}$ is said to have the weak extension property if for every algebra $\mathbf{A} \in \mathcal{K}$ and for every pair of congruences α, β of $\mathbf{A}$ such that α and β intersect to 0_A (the identity relation), the least $\mathcal{K}$-congruences of $\mathbf{A}$ including α(respectively β) also intersect to 0_A. W. Dziobiak has conjectured that every finitely generated quasivariety of finite signature with the weak extension property is finitely axiomatizable. (This is a far-reaching extension of Park's conjecture for varieties.) In this talk, I will describe some results about WEP that were proved recently by our team of W. Dziobiak, M. Maroti, A. Nurakuno, R. Willard and myself.

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When is an algebra, which is not strongly dualisable, not fully dualisable?

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Let $\underline{\mathbf{M}}$ be a dualisable (in the sense of natural duality theory) but not strongly dualisable finite algebra. We will show that, if $\underline{\mathbf{M}}$ has a special kind of dualising alter ego, then $\underline{\mathbf{M}}$ is not fully dualisable. This idea will be demonstrated on a concrete example.

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Hereditary congruence lattices

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Inspired by a paper of John Snow we introduced the concept of a hereditary congruence lattice as follows. Let L be a sublattice of the lattice of all equivalence relations on a set A. We say that L is a hereditary congruence lattice, if for every complete 0–1 sublattice L' of L there exists a set of operations F' such that $L' = Con(A; F')$. By a result of Quackenbush and Wolk every finite distributive sublattice of $Eq(A)$ is a hereditary congruence lattice. John Snow proved that every finite lattice in the lattice variety generated by M_3 is the congruence lattice of a finite algebra, by constructing a hereditary congruence lattice representation of M_3^n. In a joint paper with Pál Hegedűs we generalized this to a much larger variety of modular lattices, again by constructing certain hereditary congruence lattices, in this case as sublattices of the congruence lattice of vector spaces over the 2-element field. It is an open question whether there is a hereditary congruence lattice isomorphic to M_4.

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Complicated ternary boolean operations

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It is well-known that the binary operations on any base set X generate all finitary operations on X. Call a finitary operation on a cardinal $\kappa \geq 2$ *boolean* if it takes only values in $\{0, 1\}$. We show that the clone of all boolean operations on infinite κ contains a very "complicated" ternary operation, in the sense that it is not generated by binary boolean operations. This yields the following statement about boolean operations on finite sets: For all $k \geq 1$ there exist $n \geq 2$ and a ternary boolean operation f on n such that f is not a term of k binary boolean operations on n.

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On the subsemilattices of formular subalgebras and congruences of the lattices of subalgebras and congruences of universal algebras

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We study the questions of independence of subsemilattices of formular subalgebras (congruences) of universal algebras from the lattices of subalgebras (congruences) of these algebras.

The subalgebras $\mathcal{B} = \langle B; \sigma \rangle$ of the algebra $\mathcal{A} = \langle A; \sigma \rangle$ is open formular (parametric open formular), it there exists some quantifierfree formula $\phi(x)$ of the signature $\sigma(\phi(x, y_1, \ldots, y_n)$ and elements $b_1, \ldots, b_n \in A)$ such that $B = \{a \in A \mid \mathcal{A} \models \phi(a)\}$ $(B = \{a \in A \mid \mathcal{A} \models \phi(a, b_1, \ldots, b_n)\})$.

The down subsemilattice of all open formular (all parametric open formular) subalgebras of the lattice $Sub\,\mathcal{A}$ of all subalgebras of the algebra $\mathcal{A}$ we denote as $PSub\,\mathcal{A}$ $(PPSub\,\mathcal{A})$.

THEOREM 1. For any algebraic lattice L and its 0—1-down subsemilattices L_0, L_1 such that $L_0 \subseteq L_1$ there exists some universal algebra $\mathcal{A}$ and isomorphism φ of the lattice L on the lattice $Sub\,\mathcal{A}$ such that $\varphi(L_0) = PSub\,\mathcal{A}$, $\varphi(L_1) = PPSub\,\mathcal{A}$.

THEOREM 2. The equation $PSub\ \mathcal{A} = Sub\ \mathcal{A}$ is true for some finite algebra $\mathcal{A}$ iff all its inner isomorphisms (the isomorphisms between subalgebras of algebra $\mathcal{A}$) are its inner automorphisms (the automorphisms of subalgebras of algebra $\mathcal{A}$).

By analogy the definition of open formular subalgebra we define the concept of open formular congruences of the algebra $\mathcal{A}$. The down subsemilattice of the lattice $Con\ \mathcal{A}$ of all congruence of algebra $\mathcal{A}$ we denote as $Con\ OF\mathcal{A}$.

THEOREM 3. For any algebraic lattice L and 0—1-down subsemilattice L_0 of the lattice $0 + L$ there exists some universal algebra $\mathcal{A}$ and the isomorphism φ of the lattice $0 + L$ on the lattice $Con\ \mathcal{A}$ such that $\varphi(L_0) = Con\ OF\mathcal{A}$.

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Congruence preserving functions on distributive lattices II

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We determine a generating set of the clone of all congruence-preserving (compatible) functions on distributive lattices. We define two new types of compatible functions and show that every compatible function is a composition of functions of known types. Especially, we prove that the clone of all compatible functions is generated by its binary members.

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On congruences of algebras defined on sectionally pseudocomplemented lattices

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On any sectionally pseudocomplemented lattice can be defined a binary operation $\circ$ similar to the relative pseudocomplementation. It is also known that any finite sublattice L of a free lattice is sectionally pseudocomplemented. The congruence lattices and the congruence classes of the corresponding algebras $(L, \wedge, \vee, \circ)$ are described. We prove that the finite sublattices of a free lattice are directly indecomposable with the exception of some very particular cases.

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Commuting operations and centralizing clones

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In this talk we define the notions in the title and briefly survey the main known results. We conclude with some more recent results obtained jointly with H. Machida from Math. Hitotsubashi University, Tokyo.

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Lattices of paths in higher dimension

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Let $P(v)$ be the collection of paths in the discrete space N^n from 0 to a point $v = (v_1, \ldots, v_n)$ with the additional property that, at each step, just one coordinate increases (by 1). A path in $P(v)$ is represented by a word on the alphabet $\{a_1, \ldots, a_n\}$ such that the number of occurrences of the letter a_i is v_i, $i = 1, \ldots, n$. The exchange of contiguous letters $a_i a_j$ with $i < j$ gives rise to a rewrite system whose reflexive transitive closure is a lattice structure on $P(v)$.

The lattice $P(v)$ generalizes lattices of permutations – for $v = (1, \ldots, 1)$ – and lattices of lattice paths – for $n = 2$. An explicit description of the join dependency relation allows to state that these lattices are semi-distributive and bounded. If $v \in N^2$, then $P(v)$ is a distributive lattice. Recall that a lattice L is near-distributive if

$$x \wedge (y \vee z) = x \wedge (y \vee (x \wedge (z \vee (x \wedge y))))$$

and the dual equation hold in L. If $v \in N^3$, then $P(v)$ is not a distributive lattice, but it is near-distributive. A glance on the lattice of permutations on four elements shows that $P(v)$ need not be near-distributive if $n = 4$.

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Varieties with definable factor congruences and BFC

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A variety $\mathcal{V}$ has *Definable Factor Congruences* (DFC) if for all $A, B \in \mathcal{V}$ the kernel congruence of the canonical projection $A \times B \to A$ can be first-order defined in terms of the *central elements* [3] associated to that congruence. We give an explicit definition and prove that for every variety $\mathcal{V}$ where proper subalgebras are always nontrivial, $\mathcal{V}$ has DFC if and only if $\mathcal{V}$ has Boolean Factor Congruences (see [1],[2],[4]).

Joint work with Diego J. Vaggione.

References

[1] D. Bigelow and S. Burris, *Boolean algebras of factor congruences,* Acta Sci. Math., **54**, 11–20.

[2] R. McKenzie, G. McNulty And W. Taylor, em Algebras, Lattices, Varieties, Volume 1, The Wadsworth & Brooks/Cole Math. Series, Monterey, California.

[3] D. Vaggione, $\mathcal{V}$ *with factorable congruences and* $\mathcal{V} = \mathbf{I}\Gamma^a(\mathcal{V}_{DI})$ *imply* $\mathcal{V}$ *is a discriminator variety,* Acta Sci. Math. **62**, 359–368.

[4] R. Willard, *Varieties Having Boolean Factor Congruences,* J. Algebra, **132**, 130–153.

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Congruence lattices of algebras with permuting congruences

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We prove that there is a distributive algebraic lattice that is not isomorphic to the congruence lattice of any algebra with permutable congruences.

Joint work with Pavel Ruzicka and Fred Wehrung.

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An intersection property of subalgebras in congruence distributive varieties

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Call a subpower $\mathbf{B}$ of the algebra $\mathbf{A}$ k-complete if the projection of B onto every set of k coordinates equals A^k. It turns out that if $\mathbf{A}$ is finite and generates a congruence distributive variety then for any integer $n > 0$, the intersection of all 2-complete subalgebras of $\mathbf{A}^n$ is non-empty. If, on the other hand, $\mathbf{A}$ is idempotent and the variety generated by $\mathbf{A}$ fails to omit either of the unary or affine types from tame congruence theory then this intersection property fails in the variety.

We will relate this intersection property to the Constraint Satisfaction Problem and in particular to the notions of bounded width associated with constraint languages.

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Checking identities in algebras

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In this talk I give a brief survey, how to prove that the identity checking problem is hard for a given finite algebra. At the end we will see, that most of the proofs are ad hoc, there is no general way to prove such a result.

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Recent results on the dimension theory of lattice-related structures

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Our main notions of "dimension" are the following.

For a ring R, let $\mathrm{FP}(R)$ denote the class of finitely generated projective right R-modules. The nonstable K-theory of R is the (commutative) monoid $V(R)$ of all isomorphism classes of objects from $\mathrm{FP}(R)$, with addition defined by $[X]+[Y] = [X \oplus Y]$, where $[X]$ denotes the isomorphism class of X. So $V(R)$ is a precursor of the classical $K_0(R)$.

For a lattice L, the dimension monoid $\mathrm{Dim}L$ of L is the commutative monoid defined by generators $D(x, y)$, where $x \leq y$ in L, and relations $D(x, x) = 0$, $D(x, z) = D(x, y) + D(y, z)$ for $x \leq y \leq z$, and $D(x \wedge y, x) = D(y, x \vee y)$, for all x, y, z in L. It turns out that $\mathrm{Dim}L$ is a precursor of the congruence lattice of L.

For an algebra A, let $\mathrm{Con}A$ denote the congruence lattice of A.

It is known that those notions are very closely related, and even enrich each other in a few cases. More precisely, dimension monoids of (complemented) modular lattices are related to the nonstable K-theory of von Neumann regular rings. Nevertheless dimension monoids are also interesting in the non-modular case, for instance for join-semidistributive lattices. In particular, in the finite join-semidistributive case, lower boundedness can be read on the dimension monoid.

We shall review a few (sometimes recent) results and problems about the ranges of the functors $R \mapsto V(R)$, $L \mapsto \mathrm{Dim}L$, and the classical congruence lattice functor, with a few solutions (either positive or negative).

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The full implies strong problem for commutative rings

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In the theory of natural dualities it is still not known if every full duality is strong. I consider this question to be one of the most interesting and irritating problems of duality theory. It is known that for some finite algebras the answer to the question is yes for straightforward, "finitary" reasons. Davey, Haviar and Willard [2] gave the first example of an algebra (the three-element bounded lattice) for which the answer is yes but not for finitary reasons. This has been recently extended by Davey, Haviar and Niven [1] who answer the full-implies-strong question affirmatively for any finite bounded distributive lattice and show that the reason is not finitary whenever the lattice is not boolean.

In this lecture I continue the exploration of the question, this time in the domain of finite commutative rings.

Theorem.
(1) If p is prime and $q = p^2$, then every alter ego for the ring Z_q which fully dualizes $\mathbf{ISP}(Z_q)$ also strongly dualizes $\mathbf{ISP}(Z_q)$. The reason is not finitary if $p \neq 2$.

(2) The same claim is true when Z_q is replaced by any finite indecomposable ring $R \in \mathbf{ISP}(Z_q)$. The reason is not finitary if $|R| \neq 4$.

This is joint work with Jennifer Hyndman and Todd Niven.

References

[1] B. A. Davey, M. Haviar, and T. Niven, *When is a full duality strong?* manuscript, 2005.

[2] B. A. Davey, M. Haviar, and R. Willard, *Full does not imply strong, does it?* Algebra Universalis, to appear.

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