Recent results on the dimension theory of lattice-related structures

Friedrich WEHRUNG

Département de Mathématiques, Université de Caen

Our main notions of "dimension" are the following.

For a ring R, let $\mathrm{FP}(R)$ denote the class of finitely generated projective right R-modules. The nonstable K-theory of R is the (commutative) monoid $V(R)$ of all isomorphism classes of objects from $\mathrm{FP}(R)$, with addition defined by $[X]+[Y]=[X\oplus Y]$, where $[X]$ denotes the isomorphism class of X. So $V(R)$ is a precursor of the classical $K_0(R)$.

For a lattice L, the dimension monoid $\mathrm{Dim}L$ of L is the commutative monoid defined by generators $D(x,y)$, where $x\leq y$ in L, and relations $D(x,x)=0$, $D(x,z)=D(x,y)+D(y,z)$ for $x\leq y\leq z$, and $D(x\wedge y,x)=D(y,x\vee y)$, for all x, y, z in L. It turns out that $\mathrm{Dim}L$ is a precursor of the congruence lattice of L.

For an algebra A, let $\mathrm{Con}A$ denote the congruence lattice of A.

It is known that those notions are very closely related, and even enrich each other in a few cases. More precisely, dimension monoids of (complemented) modular lattices are related to the nonstable K-theory of von Neumann regular rings. Nevertheless dimension monoids are also interesting in the non-modular case, for instance for join-semidistributive lattices. In particular, in the finite join-semidistributive case, lower boundedness can be read on the dimension monoid.

We shall review a few (sometimes recent) results and problems about the ranges of the functors $R\mapsto V(R)$, $L\mapsto\mathrm{Dim}L$, and the classical congruence lattice functor, with a few solutions (either positive or negative).