Lattices of Paths in Higher Dimension

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Let $P(v)$ be the collection of paths in the discrete space N^n from 0 to a point $v = (v_1, \ldots, v_n)$ with the additional property that, at each step, just one coordinate increases (by 1). A path in $P(v)$ is represented by a word on the alphabet $\{a_1, \ldots, a_n\}$ such that the number of occurrences of the letter a_i is v_i, $i = 1, \ldots, n$. The exchange of contiguous letters $a_i a_j$ with $i < j$ gives rise to a rewrite system whose reflexive transitive closure is a lattice structure on $P(v)$.

The lattice $P(v)$ generalizes lattices of permutations – for $v = (1, \ldots, 1)$ – and lattices of lattice paths – for $n = 2$. An explicit description of the join dependency relation allows to state that these lattices are semi-distributive and bounded. If $v \in N^2$, then $P(v)$ is a distributive lattice. Recall that a lattice L is near-distributive if

$$x \wedge (y \vee z) = x \wedge (y \vee (x \wedge (z \vee (x \wedge y))))$$

and the dual equation hold in L. If $v \in N^3$, then $P(v)$ is not a distributive lattice, but it is near-distributive. A glance on the lattice of permutations on four elements shows that $P(v)$ need not be near-distributive if $n = 4$.