Hypergraph homomorphisms and compositions of Boolean functions with clique functions

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For a class $\mathcal{C}$ of Boolean functions, we say that a Boolean function f is a $\mathcal{C}$-*subfunction* of a Boolean function g, denoted $f \preceq_{\mathcal{C}} g$, if $f = g(h_1, \ldots, h_n)$, where all the inner functions h_i are members of $\mathcal{C}$. The $\mathcal{C}$-subfunction relation is a preorder on the set Ω of all Boolean functions if and only if $\mathcal{C}$ is a clone, and it is natural to ask whether there is an infinite descending chain of $\mathcal{C}$-subfunctions and what is the size of the largest antichain of $\mathcal{C}$-incomparable functions.

In this presentation, we focus on the clones U_k of all 1-*separating functions* (or *clique functions*) of rank k (for $k = 2, \ldots, \infty$). We characterize the U_k-subfunction relation in terms of homomorphisms between hypergraphs.

For a function f, we define the *rank-k disjointness hypergraph of f*, denoted $G(k, f)$, as follows: the vertices of $G(k, f)$ are the true points of f and $S \in E(G(k, f))$ if and only if $2 \leq |S| \leq k$ and $\bigwedge S = \mathbf{0}$. For $k = 2$, this is an ordinary graph—in fact it is the complement of the intersection graph of the true points of f—and we call it the *disjointness graph of f*. It is well-known that every graph is an intersection graph; hence every graph is the rank-2 disjointness graph of some function.

Theorem 1 *Let f and g be 0-preserving functions. Then $f \preceq_{U_k} g$ if and only if $G(k, f)$ is homomorphic to $G(k, g)$.*

For each k, there is an infinite sequence of hypergraphs $G_1^k, G_2^k, \ldots$ such that G_i^k is the rank-k disjointness hypergraph of some function and, for all i, G_{i+1}^k is homomorphic to G_i^k but G_i^k is not homomorphic to G_{i+1}^k. By Theorem 1, there is an infinite descending chain of U_k-subfunctions.

There exists also an infinite family of graphs that are pairwise incomparable in the sense that they are not homomorphic to each other. Theorem 1 implies that there exists an infinite antichain of U_2-incomparable functions and hence an infinite antichain of U_k-incomparable functions for any $k \geq 2$.