Arithmetical affine complete varieties and inverse monoids

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We call an algebra *minimal* if it has no proper subalgebras and *weakly diagonal* if every subuniverse of its square contains the graph of some of its automorphisms. It is known that every affine complete variety of finite type is generated by a finite weakly diagonal algebra which is unique up to isomorphism and that every finite minimal algebra of arithmetical variety generates affine complete variety.

Let $\mathbf{A}$ be a finite minimal algebra generating arithmetical (hence also affine complete) variety. Then the set of all non-empty subuniverses of $\mathbf{A}^2$ forms an inverse monoid with respect to the relational product operation. The elements of this monoid are called *bicongruences* of $\mathbf{A}$ and the monoid itself is denoted by $\mathbf{BiconA}$.

The algebras $\mathbf{A}$ and $\mathbf{B}$ are said to be *categoricallly equivalent* if there is an equivalence of categories $F : \mathrm{Var}\mathbf{A} \to \mathrm{Var}\mathbf{B}$ such that $F(\mathbf{A}) = \mathbf{B}$. Two varieties are called categorically equivalent if they have categorically equivalent generators.

Theorem 1. Two finite minimal algebras generating arithmetical varieties are categorically equivalent iff their monoids of bicongruences are isomorphic.

We would like to describe the monoids appearing in Theorem 1 but so far we have not been able to solve this problem. However, we were able to handle the weakly diagonal case. Note that an inverse monoid $\mathbf{S}$ is called *factorizable* if for every $s \in S$ there exists a unit g of $\mathbf{S}$ such that $s \leq g$ (here $\leq$ is the natural order relation of $\mathbf{S}$).

Theorem 2. Given a finite inverse monoid $\mathbf{S}$, the following are equivalent:

(1) there exists a finite weakly diagonal algebra $\mathbf{A}$ such that $\mathrm{Var}\mathbf{A}$ is arithmetical and $\mathbf{BiconA} \simeq \mathbf{S}$;

(2) $\mathbf{S}$ is factorizable, has a multiplicative zero, its idempotents form a distributive lattice under the natural order relation and the following condition holds:
(IU) for every two idempotents e_1, e_2 and a unit g of $\mathbf{S}$, if $e_1e_2 \leq g$ then there exist units g_1, g_2 of $\mathbf{S}$, such that $e_1 \leq g_1$, $e_2 \leq g_2$ and $g_1g_2 = g$.

This theorem yields a 1-1 correspondence between categorical equivalence classes of arithmetical affine complete varieties of finite type and isomorphism classes of monoids satisfying conditions (2) in Theorem 2.