ON THE LATTICE OF EQUATIONAL CLASSES OF OPERATIONS AND ITS MONOIDAL INTERVALS

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Let A be a finite non-empty set. By a *class* of operations on A we simply mean a subset $\mathcal{I} \subseteq \cup_{n\geq 1} A^{A^n}$. The *composition* of two classes $\mathcal{I}, \mathcal{J} \subseteq \cup_{n\geq 1} A^{A^n}$ of operations on A, denoted $\mathcal{IJ}$, is defined as the set

$$\mathcal{IJ} = \{f(g_1, \ldots, g_n) \mid n, m \geq 1, f \text{ } n\text{-ary in } \mathcal{I},\, g_1, \ldots, g_n \text{ } m\text{-ary in } \mathcal{J}\}.$$

A *functional equation* (for operations on A) is a formal expression

$$h_1(\mathbf{f}(g_1(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, \mathbf{f}(g_m(\mathbf{v}_1, \ldots, \mathbf{v}_p))) = h_2(\mathbf{f}(g'_1(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, \mathbf{f}(g'_t(\mathbf{v}_1, \ldots, \mathbf{v}_p)))$$

where $m, t, p \geq 1$, $h_1 : A^m \to A$, $h_2 : A^t \to A$, each g_i and g'_j is a map $A^p \to A$, the $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are p distinct vector variable symbols, and $\mathbf{f}$ is a function symbol. An n-ary operation f on A is said to *satisfy* the above equation if, for all $v_1, \ldots, v_p \in A^n$, we have

$$h_1(f(g_1(v_1, \ldots, v_p)), \ldots, f(g_m(v_1, \ldots, v_p))) = h_2(f(g'_1(v_1, \ldots, v_p)), \ldots, f(g'_t(v_1, \ldots, v_p)))$$

The classes of operations definable by functional equations (*equational classes*) are known to be exactly those classes $\mathcal{I}$ satisfying $\mathcal{IO}_A = \mathcal{I}$, where $\mathcal{O}_A$ denotes the class of all projections on A (for $|A| = 2$ see [3, 4], and for $|A| \geq 2$ see [1]). In particular, clones of operations, i.e. classes containing all projections and idempotent under class composition, are equational classes.

The set of all equational classes on A constitute a complete lattice under union and intersection. Moreover, it is partially ordered monoid under class composition, with identity $\mathcal{O}_A$, and whose non-trivial idempotents are exactly the clones on A. But the classification of operations into equational classes is much finer than the classification into clones: for $|A| = 2$, there are uncountably many equational classes on A, but only countably many of them are clones. Also, the set of clones does not constitute a monoid since it is not closed under class composition (see [2]).

The aim of this presentation is to investigate the lattice of equational classes of operations on a finite set A. For $|A| = 2$, we classify all monoidal intervals $[\mathcal{C}_1, \mathcal{C}_2]$, for clones $\mathcal{C}_1$ and $\mathcal{C}_2$, in terms of their size: we give complete descriptions of the countable intervals, and provide families of uncountably many equational classes in the remaining intervals. In particular, from this classification it will follow that an interval $[\mathcal{C}_1, \mathcal{C}_2]$ contains uncountably many equational classes if and only if $\mathcal{C}_2 \setminus \mathcal{C}_1$ contains a "non-associative" Boolean function.

References

[1] M. Couceiro, S. Foldes. "Constraints, Functional Equations, Definability of Function Classes, and Functions of Boolean Variables", Rutcor Research Report 36-2004, Rutgers University, `http://rutcor.rutgers.edu/~rrr/`.

[2] M. Couceiro, S. Foldes, E. Lehtonen. "Composition of Post Classes and Normal Forms of Boolean Functions", Rutcor Research Report 05-2005, Rutgers University, `http://rutcor.rutgers.edu/~rrr/`.

[3] O. Ekin, S. Foldes, P. L. Hammer, L. Hellerstein. "Equational Characterizations of Boolean Functions Classes", *Discrete Mathematics*, **211** (2000) 27–51.

[4] N. Pippenger. "Galois Theory for Minors of Finite Functions", *Discrete Mathematics,* **254** (2002) 405–419.

[5] E. L. Post. "The Two-Valued Iterative Systems of Mathematical Logic", *Annals of Mathematical Studies* **5** (1941) 1–122.

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