Semilattice ordered algebras I
Free algebras

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The operation ω **distributes over** $+$ means that for any $x_1, \ldots, x_i, y_i, \ldots, x_n \in A$

$$\omega(x_1, \ldots, x_i + y_i, \ldots, x_n) = \\ \omega(x_1, \ldots, x_i, \ldots, x_n) + \omega(x_1, \ldots, y_i, \ldots, x_n),$$

for any $1 \leq i \leq n$.

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2. $\omega(x_{11}, \ldots, x_{n1}) + \ldots + \omega(x_{1r}, \ldots, x_{nr}) \leq \omega(x_{11} + \ldots + x_{1r}, \ldots, x_{n1} + \ldots + x_{nr})$.

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3. $\omega(x, \ldots, x) \leq x$ iff $\omega(x_1, \ldots, x_n) \leq x_1 + \ldots + x_n$.

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An algebra (M, Ω) is called a **mode** if it is **idempotent** and **entropic**:

$$\omega(x, \dots, x) = x, \qquad \text{(idempotent law)},$$

$$\omega(\phi(x_{11}, \dots, x_{n1}), \dots, \phi(x_{1m}, \dots, x_{nm})) =$$

$$\phi(\omega(x_{11}, \dots, x_{1m}), \dots, \omega(x_{n1}, \dots, x_{nm})), \qquad \text{(entropic law)},$$

for every m-ary $\omega \in \Omega$ and n-ary $\phi \in \Omega$.

- For semilattice ordered modes one has

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  Let $\mathcal{P}_{>0}(A)$ be the family of all non-empty subsets of a given set A. For any n-ary operation $\omega : A^n \to A$ we define **the complex operation** $\omega : \mathcal{P}_{>0}(A)^n \to \mathcal{P}_{>0}(A)$ in the following way:

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  The algebra $(\mathcal{P}_{>0}^{<\omega}A, \Omega, \cup)$ of all finite non-empty subsets of A is a subalgebra of $(\mathcal{P}_{>0}A, \Omega, \cup)$.

Let $(F_{\mathcal{V}}(X), \Omega)$ be the free algebra over a set X in the variety $\mathcal{V} \subseteq \mho$ and let $\mathcal{S}_{\mathcal{V}}$ denote the variety of all semilattice ordered $\mathcal{V}$-algebras.

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Theorem (Universality Property for Semilattice Ordered Algebras)

Let $(A, \Omega, +) \in \mathcal{S}_{\mathcal{V}}$. Each mapping $h\colon X \to A$ can be extended to a unique homomorphism $\overline{\overline{h}}\colon (\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}}(X), \Omega, \cup) \to (A, \Omega, +)$, such that $\overline{\overline{h}}/_X = h$.

Remarks

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Corollary

The semilattice ordered algebra $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}}(X), \Omega, \cup)$ is free over a set X in the variety $\mathcal{S}_{\mathcal{V}}$ if and only if $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}}(X), \Omega, \cup) \in \mathcal{S}_{\mathcal{V}}$.

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G. Grätzer and H. Lakser proved that for any subvariety $\mathcal{V} \subseteq \mho$, the algebra $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}}(X), \Omega)$ satisfies the identities being a result of identification of variables from the linear identities true in $\mathcal{V}$. This implies that for each subvariety $\mathcal{V} \subseteq \mho$, the algebra $(\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}}(X), \Omega)$ belongs to $\mathcal{V}^*$, but it does not belong to any its proper subvariety.

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Corollary

For any subvariety $\mathcal{V} \subseteq \mathcal{U}$, *we have*

$$\mathcal{S}_{\mathcal{V}^*} = \mathrm{HSP}((\mathcal{P}^{<\omega}_{>0}F_{\mathcal{V}^*}(X), \Omega, \cup)),$$

for a non-finite set X.