Modular and maximal chains in the subgroup lattice of a finite group

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Joint work with John Shareshian.

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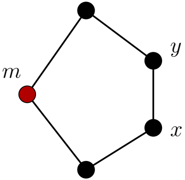		

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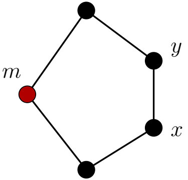	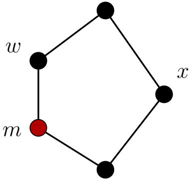	

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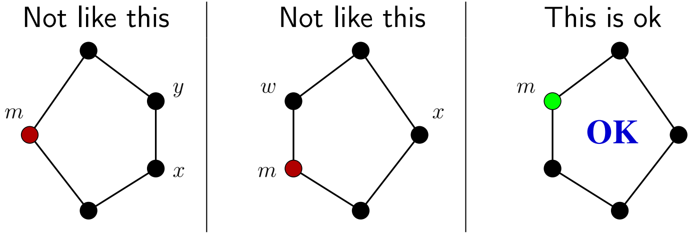

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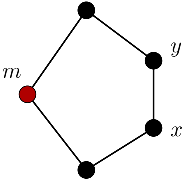

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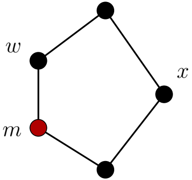

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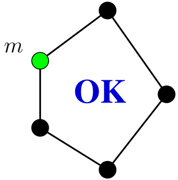

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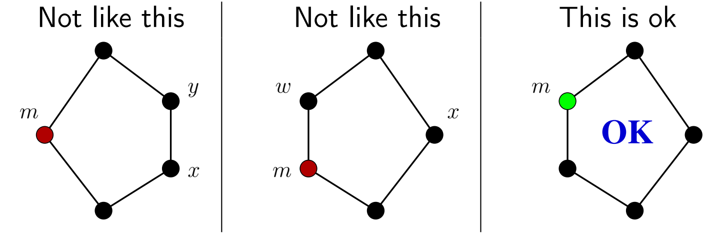

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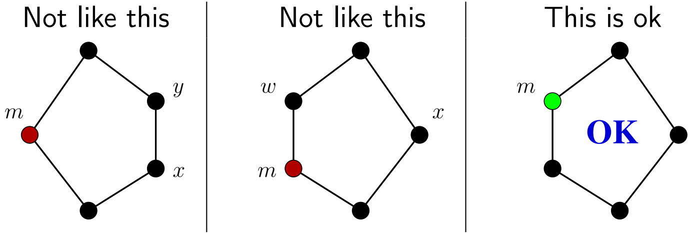

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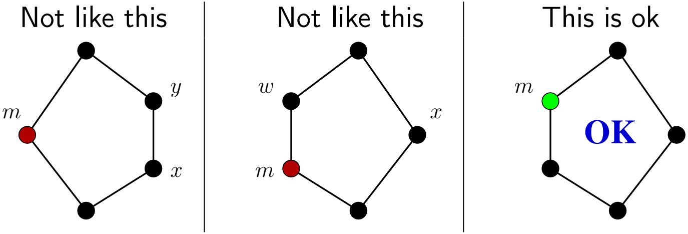

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Not every modular subgroup is normal.

A bigger example

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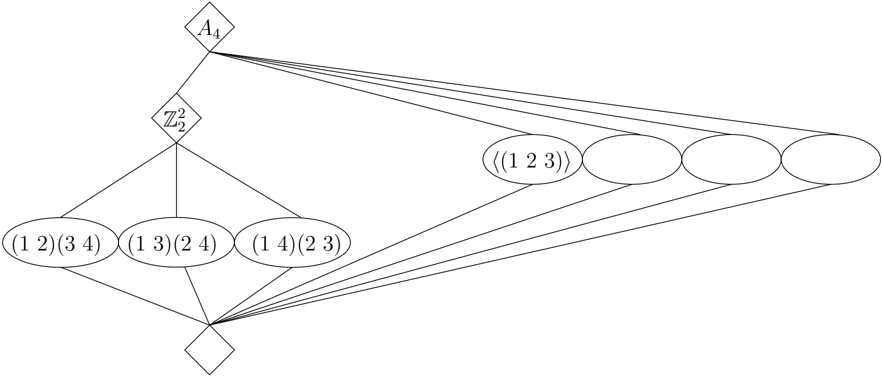

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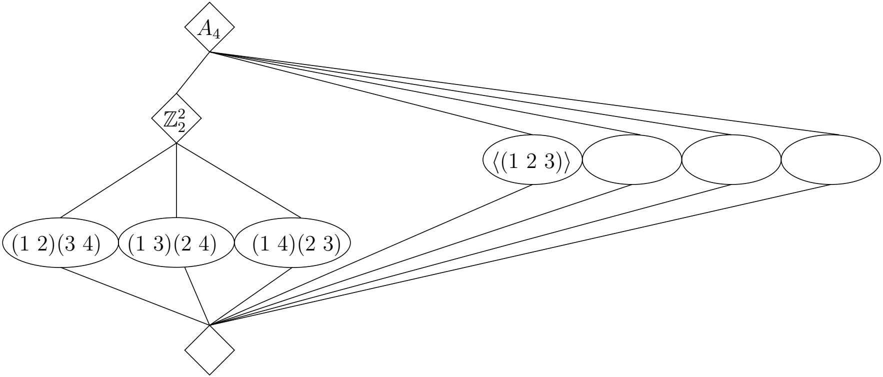

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(In fact, every maximal chain has the same length.)

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An easy special case of the maximum length modular chain theorem

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How does this compare with other such characterizations?

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Our result has a considerably different character from Schmidt's.

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We regard our result as being an Iwasawa-type characterization of solvable groups.

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Thm 2: G is solvable $\iff$
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(This is a quite hard question even in p-groups!)

References:

John Shareshian and Russ Woodroofe, *A new subgroup lattice characterization of finite solvable groups*, arXiv:1011.2503.

Thank you!

Russ Woodroofe
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Example: $L(S_4)$

Thm 1: max length modular chain = length chief series.

Thm 2: G is solvable $\iff$
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$L(S_4)$, the symmetric group on 4 elements: