On derived weak congruence representability

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Introduction

Derived weak congruence representability is a brand new direction in the investigation of the weak congruence representability, where the representability of a lattice is derived from the representability of another lattice, or some set of lattices. We show that in certain cases an interval sublatice or another sublattice or a suborder of a representable lattice is representable. Starting from a representation of the lattice we build a representation of the mentioned related lattice. In a similar way, two cases when the representability of some set of lattices implies the representability of another lattice are given.

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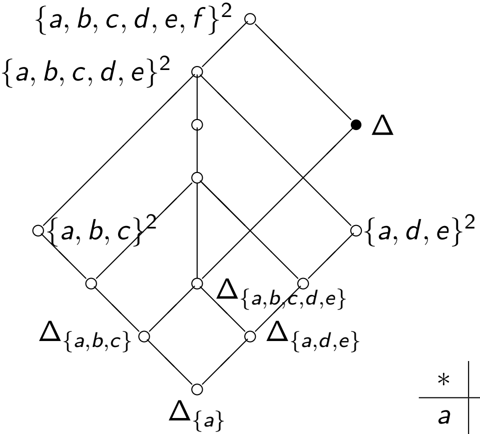
All the weak congruences of an algebra form a lattice under inclusion, which is called the weak congruence lattice. This is an algebraic lattice.



$*$	a	b	c	d	e	f
a	a	c	b	e	d	b
b	b	c	b	b	c	e
c	c	c	b	c	b	f
d	e	d	e	e	d	d
e	d	e	d	d	d	c
f	d	f	e	c	b	e

α	a	b	c	d	e	f
	a	a	a	a	a	b

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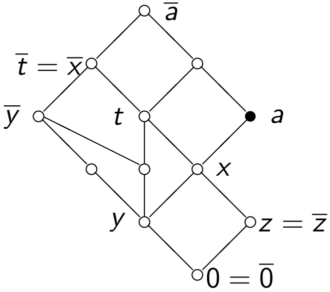

Theorem

If a lattice L is weak congruence representable and $a \in L$ corresponds to the diagonal relation of the representing algebra, then also the interval sublattice $[y, \overline{x}]$ of L is representable, element $a \wedge \overline{x}$ for all $x, y \in L$, $y \leqslant x \leqslant a$.

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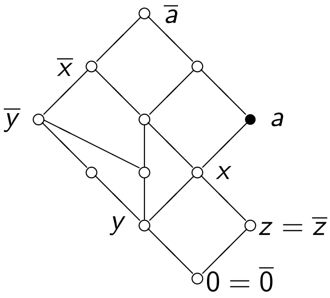

Derived representability

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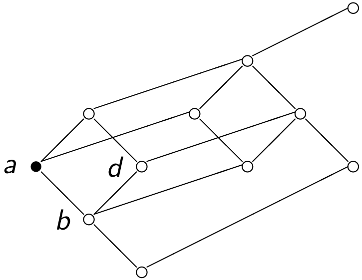
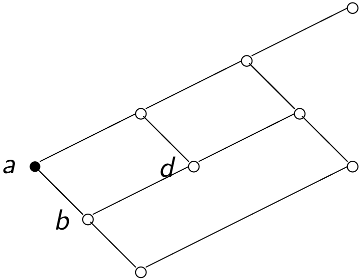
This may not be a sublattice of the initial lattice, except under some conditions.



lattice L



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If a is Δ-suitable element of a lattice L and $d \in \uparrow a$. Set $L' = \downarrow d \cup \{\bar{b} \mid b \leqslant a\}$ is a lattice under the order in L, element a being a Δ-suitable in the lattice.

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(i) If there are at least two lattices $L_i, L_j (i, j \in I, i \neq j)$ that are, together with their elements a_i, a_j, represented by the weak congruence lattices of algebras, each having at least one constant, then for every $b \in L$, $b \leqslant a = (a_i)_{i \in I}$ we add another element b', such that $b' \wedge a = b$ and b' is greater from all elements of the set $\{x \in L \mid x \wedge a = b\}$, and the following inequalities hold:

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$x \leqslant b' \Leftrightarrow x \wedge a \leqslant b$;
$b' \leqslant x \Leftrightarrow (x = c' \wedge c \geq b)$.

(ii) If there exists only one lattice L_j in the set of lattices $\{L_i \mid i \in I\}$ that can be represented, together with element a_j, by the week congruence lattice of an algebra with at least one constant, then we take lattice L' as in case (i), without elements of the form l', where $l = (l_i)_{i \in I}$, $l_j \leqslant a_j$ and $l_i = 0$ whenever $i \neq j$.

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Theorem

If a_i is a Δ-suitable element of a lattice L_i, for all $i \in I$, then $a = (a_i)_{i \in I}$ is Δ-suitable in the extended direct product of the family $\{(L_i, a_i) \mid i \in I\}$.

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$L' = L \cup S$, where $S \cap L = \emptyset$, $S = \{s_b \mid b \in L, b \leqslant a\}$. Now we define an order $\leqslant'$ on L' as follows:

If $x, y \in L$, then $x \leqslant' y$ if and only if $x \leqslant y$.

If $x, y \in S$, and $x = s_b$, $y = s_c$, $b, c \in L$, then $x \leqslant' y$ if and only if $b \leqslant c$.

If $x \in L$, $y \in S$, and $y = s_b$, then $y \nleqslant' x$, and $x \leqslant' y$ if and only if $x \wedge a \leqslant b$.

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We call this extension $\downarrow a$-extension of the initial lattice.

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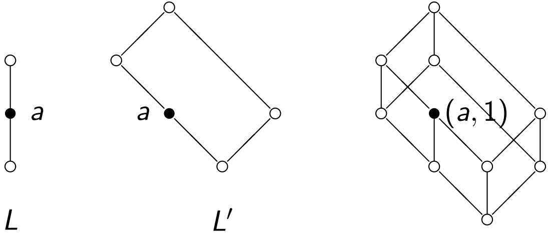
Let L be a weak congruence representable lattice and let $a \in L$ corresponds to the diagonal of the algebra representing L. Let L' be the $\downarrow a$-extension of L.
If M is any algebraic lattice, then $L' \times M$ is weak congruence representable and the element corresponding to the diagonal of the representing algebra is $(a, 1)$.



a
a
$(a,1)$
L
L'

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$0 \in L$ is Δ-suitable: take an algebra $\mathcal{A}$, such that $Con\mathcal{A} \cong L$, and add all its elements, as constant operations, to the set of its operations. Thus we get $\mathcal{A}'$, whose only subalgebra is $\mathcal{A}'$ itself. Since the weak congruences of an algebra are congruences on subalgebras, we get $Cw\mathcal{A}' = Con\mathcal{A}' = Con\mathcal{A} \cong L$. Applying the previous theorem to L', the $\downarrow$0-extension of L, we get the following:

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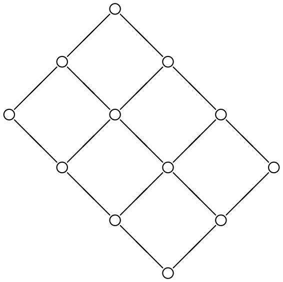
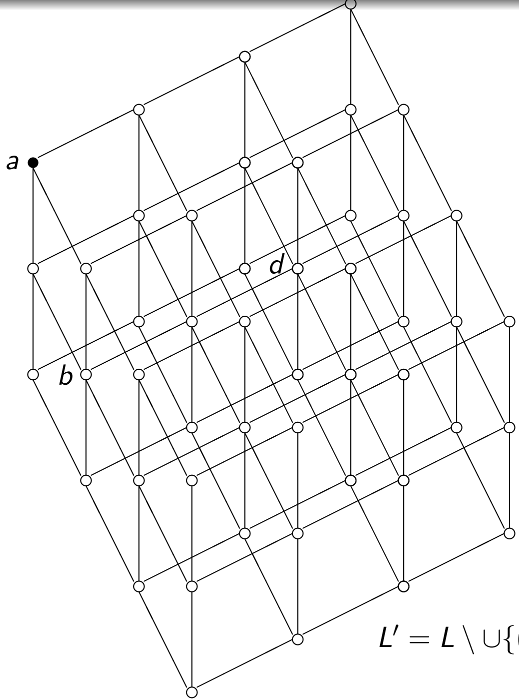
(A. Tepavčević) If L be an algebraic lattice and $a \in L$ an element from the center of the lattice, such that $\uparrow a = \downarrow b \cup \{1\}$, for some $b \in L$, a is Δ-suitable.

L

L'



M



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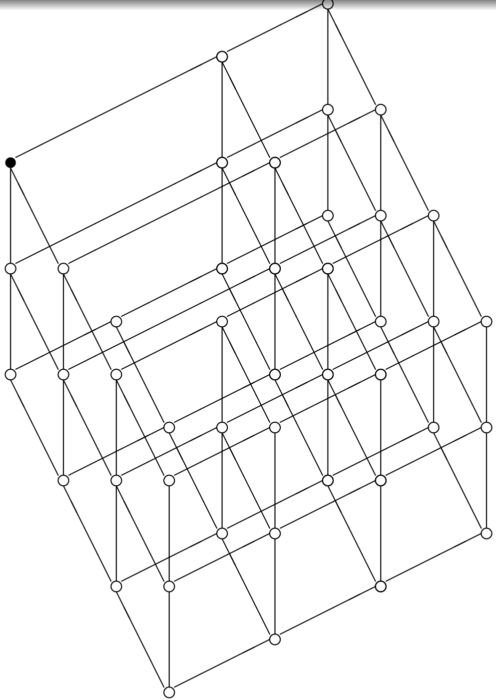

Thank you for your attention!