Universal Algebra and Lattice Theory

Dedicated to the 80th birthday of Béla Csákány

Lattice valued identities and equational classes

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Usually a lattice L being the co-domain is known and fixed. Therefore we say that these are ***L*-valued structures**, or ***L*-structures**, for short.
Special L-valued equalities are introduced in order to replace the ordinary classical equality in dealing with L-valued identities.

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We prove that if an L-valued subalgebra of an algebra satisfies an L-valued identity with respect to some L-valued equality, then there is a least L-equality such that the corresponding L-identity holds on the same L-valued subalgebra.

Abstract cont.

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Next we introduce and investigate L-valued equational classes. These are defined with respect to a set of lattice valued identities, and consist of L-valued algebras of the same type, fulfilling all L-identities in the given set. In this lattice valued framework we introduce basic notions of universal algebra: lattice valued homomorphisms (H), lattice valued subalgebras (S), and lattice valued direct products (P). Our main result is that every lattice valued equational class is closed under these three constructions (H, S and P), hence forming a lattice valued variety.

Abstract cont.

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The notion of a fuzzy equality was introduced by Höhle in 1988 and then used by many others. Demirci (1999–2003), considers particular algebraic structures equipped with a fuzzy equality relation; he also uses compatible fuzzy functions. Bělohlávek (the book from 2002, also with Vychodil 2005–6) introduces and investigates algebras with fuzzy equalities. These are defined as classical algebras in which the crisp equality is replaced by a fuzzy one being compatible with the fundamental operations of the algebra. Bělohlávek develops and investigates the most important fuzzified universal algebraic topics in this framework.

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Consequently, a mapping $\rho : A^2 \to L$ is a **lattice valued** (binary) **relation** on A.

If $\mu : A \to L$ is an L-valued set on a nonempty set A, then a lattice valued relation $\rho : A^2 \to L$ on A is said to be a **lattice valued relation on μ** if for all $x, y \in A$

$$\rho(x, y) \leqslant \mu(x) \wedge \mu(y).$$

A lattice valued relation ρ on a lattice valued set μ is **reflexive** if for all $x, y \in A$,

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Lemma

If ρ is a reflexive lattice valued relation on a lattice valued set μ on A, then for every $x, y \in A$,

$$\rho(x, x) \geqslant \rho(x, y) \ \text{ and } \ \rho(x, x) \geqslant \rho(y, x).$$

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A reflexive, symmetric and transitive relation ρ on a lattice valued set μ is a **lattice valued equivalence** on μ.

A lattice valued equivalence relation ρ on μ, fulfilling for all $x, y \in A$, $x \neq y$,:

$$\text{if } \rho(x, x) \neq 0, \text{ then } \rho(x, x) > \rho(x, y) \ \text{ and } \ \rho(x, x) > \rho(y, x),$$

is called a **lattice valued equality** relation on μ.

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For a nullary operation (constant) $c \in F$, we require that

$$\mu(c) = 1,$$

where 1 is the greatest (the top) element in L.

An L-valued relation $\rho : A^2 \to L$ on an L-subalgebra $\mu : A \to L$ of $\mathcal{A} = (A, F)$ is said to be **compatible** with the operations, if for every n-ary operation $f \in F$ and for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$

$$\rho(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \geqslant \bigwedge_{i=1}^{n} \rho(x_i, y_i).$$

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Obviously, particular L-valued congruences on μ are compatible L-valued equalities on this L-subalgebra of $\mathcal{A}$.

Lattice valued identity

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A lattice valued **identity** of the type (F, σ) over a set of variables X is the expression $E(t_1, t_2)$, where $t_1(x_1, \dots, x_n), t_2(x_1, \dots, x_n)$, briefly t_1, t_2 belong to the set $T(X)$ of terms over X and both have at most n variables.

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An L-valued subalgebra μ of an algebra $\mathcal{A} = (A, F)$ **satisfies** a lattice valued identity $E(t_1, t_2)$ (this identity **holds** on μ) with respect to L-equality E_i on $\mathcal{A}$, if for all $x_1, \ldots, x_n \in A$

$$\bigwedge_{i=1}^{n} \mu(x_i) \leqslant E_i(t_1, t_2).$$

Theorem

Let μ be an L-valued subalgebra of $\mathcal{A}$, such that there is an L-equality E on μ so that μ satisfies identity $E(f,g)$ for terms f,g in the language of $\mathcal{A}$. Then there is the least L-valued equality on μ, denoted by $E_{\mu(f,g)}$, such that μ satisfies $E_{\mu(f,g)}(f,g)$.

Lattice valued algebra

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Let $\mathcal{A} = (A, F_A)$ be an algebra, let $\mu_A : A \to L$ be a lattice valued subalgebra of $\mathcal{A}$ and $E_A : A^2 \to L$ a compatible lattice valued equality on μ_A.

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Then, $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ is a **lattice valued algebra** of the type (F, σ).

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In other words, a *lattice valued algebra is an L-valued subalgebra of a given algebra, endowed with a compatible lattice valued equality.*

An L-valued algebra $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ **satisfies** an L-identity $E(t_1, t_2)$ (this identity **holds** on $\bar{\mathcal{A}}$), if for all $x_1, \dots, x_n \in A$

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Equational class

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Equational class

Let Σ be a set of lattice valued identities of the type (F, σ) and let L be a fixed complete lattice. Then all L-valued algebras $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ of this type satisfying all identities in Σ form an **equational class** $\mathfrak{M}$ of L-valued algebras.

Operator S

Operator S

Theorem

Let $\bar{A} = (\mathcal{A}, \mu_A, E_A, L)$ be an L-valued algebra and $\mu_B : A \to L$ an L-subalgebra of $\mathcal{A}$, fulfilling the following conditions:

1. $\mu_B(x) \leqslant \mu_A(x)$ *for all* $x \in A$.
2. *If x and y are distinct elements of A and $\mu_B(x) > 0$, then* $E_A(x, y) < \mu_B(x)$.
3. $\mu_B(c) = \mu_A(c)$ *, for any constant c in the language.*

Then, a lattice valued relation E_B on μ_B given by

$$E_B(x, y) := E_A(x, y) \wedge \mu_B(x) \wedge \mu_B(y),$$

is a compatible lattice valued equality on μ_B.

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3. $\mu_B(c) = \mu_A(c)$, for any constant c.
4. $E_B(x, y) := E_A(x, y) \wedge \mu_B(x) \wedge \mu_B(y)$.

Then we say that the L-valued algebra $\bar{\mathcal{B}} = (\mathcal{A}, \mu_B, E_B, L)$ is a (**lattice valued**) **subalgebra** of $\bar{\mathcal{A}}$.

Theorem

Let $\mathfrak{M}$ be an equational class of L-valued algebras and let $\bar{\mathcal{A}} \in \mathfrak{M}$ where $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$. If $\bar{\mathcal{B}} = (\mathcal{A}, \mu_B, E_B, L)$ is an L-valued subalgebra of $\bar{\mathcal{A}}$, then also $\bar{\mathcal{B}} \in \mathfrak{M}$.

Operator H

Operator H

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ and $\bar{\mathcal{B}} = (\mathcal{B}, \mu_B, E_B, L)$ be lattice valued algebras of the same type. We say that $f : A \to B$ is a **lattice valued mapping** of $\bar{\mathcal{A}}$ into $\bar{\mathcal{B}}$ if the following conditions hold:
1. $(\forall a \in A)\ \mu_B(f(a)) \geqslant \mu_A(a)$
2. Let $t_1(x_1, ..., x_n)$, $t_2(x_1, ..., x_n)$ be terms in the language of $\mathcal{A}$, let t_1^A, t_2^A be the corresponding term operations and $a_1, ..., a_n$ elements from A.

$$\text{If} \quad E_A(t_1^A(a_1, ..., a_n), t_2^A(a_1, ..., a_n)) \geqslant \bigwedge_{i=1}^{n} \mu_A(a_i),$$

$$\text{then} \quad E_B(f(t_1^A(a_1, ..., a_n)), f(t_2^A(a_1, ..., a_n))) \geqslant$$

$$\mu_B(f(t_1^A(a_1, ..., a_n))) \wedge \mu_B(f(t_2^A(a_1, ..., a_n))).$$

Theorem

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$, $\bar{\mathcal{B}} = (\mathcal{B}, \mu_B, E_B, L)$ and $\bar{\mathcal{C}} = (\mathcal{C}, \mu_C, E_C, L)$ be L-valued algebras of the same type and $f : A \to B$, $g : B \to C$ an L-valued mappings. Then also their composition $f \circ g : A \to C$ is an L-valued mapping.

Theorem

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Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ and $\bar{\mathcal{B}} = (\mathcal{B}, \mu_B, E_B, L)$ be L-valued algebras of the same type. Then an L-valued mapping $h : A \to B$ is a **lattice valued homomorphism** of $\bar{\mathcal{A}}$ into $\bar{\mathcal{B}}$ if the following holds:

1. For each n-ary operation f_A and for all $a_1, ..., a_n \in A$, $h(f_A(a_1, ..., a_n)) = f_B(h(a_1), ..., h(a_n))$.
2. $h(c_A) = c_B$, for every nullary operation c in the language, c_A and c_B being the corresponding constants in $\mathcal{A}$ and $\mathcal{B}$ respectively.

Proposition

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ *be an* L*-valued algebra and* $\mathcal{B} = (B, F_B)$ *a subalgebra of* $\mathcal{A}$. *If*

$$\mu_B(x) := \begin{cases} \mu_A(x), & x \in B \\ 0, & else \end{cases} \quad and$$

$$E_B(x, y) := E_A(x, y) \wedge \mu_B(x) \wedge \mu_B(y),$$

then $\bar{\mathcal{B}} = (\mathcal{A}, \mu_B, E_B, L)$ *is an* L*-valued subalgebra of* $\bar{\mathcal{A}}$.

Theorem

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$$\mu_D(d) := \begin{cases} \mu_B(d), & d \in D = h(A) \\ 0, & \text{otherwise} \end{cases}$$

and

$$E_D(x, y) = E_B(x, y) \wedge \mu_D(x) \wedge \mu_D(y).$$

Then, $\bar{\mathcal{D}}$ is an L-valued subalgebra of $\bar{\mathcal{B}}$.

Theorem

Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu_A, E_A, L)$ and $\bar{\mathcal{B}} = (\mathcal{B}, \mu_B, E_B, L)$ be L-valued algebras and $h : A \to B$ an L-valued homomorphism. If $F(x_1, ..., x_n)$ is a term in the same language and F^A, F^B the corresponding term operations in $\mathcal{A}$ and $\mathcal{B}$ respectively, then h is an L-valued homomorphism of $(\bar{\mathcal{A}}, F^A)$ into $(\bar{\mathcal{B}}, F^B)$.

Theorem

Let $\mathfrak{M}$ be an equational class of lattice valued algebras. If $\bar{\mathcal{A}} \in \mathfrak{M}$ and $\bar{\mathcal{D}}$ is a homomorphic image of $\bar{\mathcal{A}}$, then also $\bar{\mathcal{D}} \in \mathfrak{M}$.

Operator P

Operator P

Theorem

Let $\{\bar{\mathcal{A}}_i = (\mathcal{A}_i, \mu_i, E_{A_i}) \mid i \in I\}$ *be a family of L-valued algebras of the same type,* $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$ *the direct product of algebras* $\mathcal{A}_i$ *and let the following holds for all* $g_1, g_2 \in \prod_{i\in I} \mathcal{A}_i$*:*

If $g_1 \neq g_2$ *and* $\bigwedge_{i\in I} \mu_i(g_1(i)) \neq 0$*, then*

$$\bigwedge_{i\in I} E_{A_i}(g_1(i), g_2(i)) \neq \bigwedge_{i\in I} \mu_i(g_1(i)).$$

Then the following holds: If

1. $\mu(g) := \bigwedge_{i \in I} \mu_i(g(i)),\ g \in \prod_{i \in I} A_i$ *and*
2. $E_A(g_1, g_2) := \bigwedge_{i \in I} E_{A_i}(g_1(i), g_2(i));\ g_1, g_2 \in \prod_{i \in I} A_i,$

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then $\bar{A} = \prod_{i\in I} \bar{A}_i := (\mathcal{A}, \mu, E_A, L)$ *is an L-valued algebra.*

The above lattice valued algebra $\bar{A} := (\mathcal{A}, \mu, E_A, L)$ is the **direct product** of L-valued algebras $\bar{A}_i, i \in I$.

Theorem

If an L-valued identity $E(u(x_1, ..., x_n), v(x_1, ..., x_n))$ *holds in all lattice valued algebras* $\bar{\mathcal{A}}_i$, $i \in I$ *of a fixed type, then also this identity holds in their product* $\bar{\mathcal{A}} = \prod_{i \in I} \bar{\mathcal{A}}_i$.

Theorem

Let $\mathfrak{M}$ be an equational class of lattice valued algebras. Then the following hold:

1. *If $\bar{\mathcal{A}} \in \mathfrak{M}$, and $\bar{\mathcal{B}}$ is a lattice valued subalgebra of $\bar{\mathcal{A}}$, then $\bar{\mathcal{B}} \in \mathfrak{M}$.*
2. *If $\bar{\mathcal{A}} \in \mathfrak{M}$, and $\bar{\mathcal{D}}$ is a homomorphic image of $\bar{\mathcal{A}}$, then $\bar{\mathcal{D}} \in \mathfrak{M}$.*
3. *If for every $i \in I, \bar{\mathcal{A}}_i$ belongs to $\mathfrak{M}$, then also $\prod_{i \in I} \bar{\mathcal{A}}_i \in \mathfrak{M}$.*

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- L-valued subalgebras of algebras mentioned above, fulfilling given identities with respect to L-valued equalities.
- L-valued algebras being L-valued subalgebras of classical algebras in the given language, such that the following holds: the algebras do not fulfill given identities, while the L-valued ones do (with respect to some L-valued equalities).

Let us describe the equational class of *L-valued groups*, with the membership values lattice given in Figure 1.

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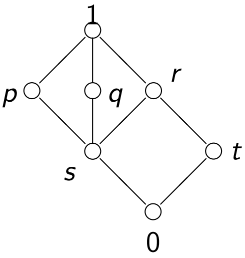

Figure 1: Lattice L

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L-valued identities defining the equational class of L-valued groups are

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This equational class consists of:

- all groups;
- L-valued subgroups of groups, as known in the theory of L-valued algebras;

The language $\mathcal{L}$ contains one binary and one unary operation (denoted by $\cdot$ and $^{-1}$ respectively), and a nullary operation (constant) e.
L-valued identities defining the equational class of L-valued groups are

$$E(x\cdot(y\cdot z),(x\cdot y)\cdot z),\ E(x\cdot e,x),E(e\cdot x,x),\ E(x\cdot x^{-1},e),\ E(x^{-1}\cdot x,e).$$

This equational class consists of:

- all groups;
- L-valued subgroups of groups, as known in the theory of L-valued algebras;
- L-valued algebras which are L-valued subalgebras of algebras in the language $\mathcal{L}$, fulfilling the above L-valued identities.

An example of such L-valued subalgebra is given in the sequel.

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$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	b	e	a
c	c	b	a	e

Table 1: Algebra G

The above algebra is not a group (associativity is not satisfied).

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Its L-valued subalgebra $\mu : G \to L$ given by

$$\mu(x) = \begin{pmatrix} e & a & b & c \\ 1 & p & q & r \end{pmatrix},$$

fulfils the above identities with respect to the L-valued equality E^{μ}:

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E^μ	e	a	b	c
e	1	s	s	s
a	s	p	s	s
b	s	s	q	s
c	s	s	s	r

Table 2: L-valued equality on μ

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Table 2: L-valued equality on μ

Hence, (G, μ, E^{μ}, L) is an L-valued group in this equational class.

Thank you for your attention!