The minimal clones generated by semiprojections on a four-element set

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Clones

$$\mathbf{k} := \{0, 1, \ldots, k-1\}$$

$$P_k := \{f \mid f : \mathbf{k}^n \to \mathbf{k}, n \geq 1\} \quad \text{(the set of all functions on } \mathbf{k}\text{)}$$

The n-ary projection onto the i-th coordinate e_i^n is defined by $e_i^n(x_1, \ldots, x_n) := x_i$ with $i \in \{1, \ldots, n\}$.

$$J_k := \{e_i^n \in P_k \mid 1 \leq i \leq n\} \quad \text{(the set of all projections on } \mathbf{k}\text{)}$$

Let $f \in P_k^n$ and $g_1, \ldots, g_n \in P_k^m$. The *composition* $f(g_1, \ldots, g_m) \in P_k^m$ is defined by

$$f(g_1, \ldots, g_n)(x) := f(g_1(x), \ldots, g_n(x))$$

for all $x \in \mathbf{k}^m$.

The set $F \subseteq P_k$ is a *clone* iff F is composition-closed and $J_k \subseteq F$.

Minimal clones

A clone $M \subseteq P_k$ is a *minimal clone*, if $M \neq J_k$ and there is no clone $C \subseteq P_k$ with $J_k \subset C \subset M$.
One important property:

Lemma

Let $M \subseteq P_k$ be a minimal clone. Then M is generated by every non-trivial member, i.e. $M = [f]$ for all $f \in M \setminus J_k$.

$[f]$ is the least clone containing f / closure of $\{f\}$.

Rosenberg's Classification Theorem

Theorem

Let f be a nontrivial operation of minimum arity in a minimal clone. Then f satisfies one of the following conditions:

(I) *f is unary, and $f^p(x) = x$ for some prime p;*

(II) *f is a binary idempotent operation, i.e. $f(x, x) = x$;*

(III) *f is a ternary majority operation, i.e. $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$;*

(IV) *f is a ternary minority operation with $f(x, y, z) = x + y + z$, where $(\mathbf{k}, +)$ is an elementary 2-group;*

(V) *f is a semiprojection, i.e. there exists an i $(1 \leq i \leq n)$ such that $f(x_1, \ldots, x_n) = x_i$ whenever the values of $x_1, \ldots, x_n$ are not pairwise distinct.*

Known results for $k = 4$

(I)	(unary)	53	(Pöschel, Kaluzhnin, 1979; precise description and formula for arbitrary k)
(II)	(binary)	2182	(Szczepara, 1995)
(III)	(majority)	2441	(Waldhauser, 2000)
(IV)	(minority)	1	(Klein 4-group)

There is a description of minimal clones generated by k-ary conservative semiprojections on $\mathbf{k}$. (Jezek, Quackenbush, 1995)

Semiprojections

Definition (Projection)

A function $f \in P_k$ is a projection, if there exists an i $(1 \leq i \leq n)$ such that $f(x_1, \ldots, x_n) = x_i$ for all values of $x_1, \ldots, x_n$.

Definition (Semiprojection)

A function $f \in P_k$ is a semiprojection, if there exists an i $(1 \leq i \leq n)$ such that $f(x_1, \ldots, x_n) = x_i$ whenever the values of $x_1, \ldots, x_n$ are not pairwise distinct.

Let's fix $i = 1$.

Are there many semiprojections on a 4-element set?

- There are $4^{24} = 2^{48} \approx 10^{15}$ ternary semiprojections.
  The same number for 4-ary semiprojections.
- How long does it take to check all of them (brute-force-style)?
- Let's be (too) optimistic: only 10000 cpu ticks per semiprojection is needed to decide if it generates a minimal clone.
- That makes over 45 CPU-years with a 2 GHz CPU.

Closure for a specific arity

Let $F \subseteq P_k$ and $n \in \mathbb{N}$. Let F^n be the n-ary functions of F.
Let $F \subseteq P_k^n$ and $m \in \mathbb{N}$. We define

$$F_{m,0} := J_k^m \quad \text{and}$$
$$F_{m,j+1} := F_{m,j} \cup \{f(g_1, \dots, g_n) \mid f \in F, g_1, \dots, g_n \in F_{m,j}\}$$

Lemma

*Let $n, m \in \mathbb{N}$ and $F \subseteq P_k^n$.
Then $[F]^m = F_{m,l}$ for some $l \in \mathbb{N}$.*

Interaction between semiprojections of different arities

Lemma

Let $f \in P_k^n$ be a semiprojection. Then $[f]$ is a minimal clone in P_k if and only if

- *there is no non-trivial n-ary semiprojection $g \in [f]^n$ with $[g]^n \subsetneq [f]^n$, and*
- *there is no non-trivial m-ary semiprojection $h \in [f]$ with $n < m \leq k$.*

Note: for $n = k$ the second condition is trivially true.

“Smaller” representative

Let f be an ternary semiprojection on $\mathbf{4}$. Then $(f(0,1,2), f(0,1,3), f(0,2,1), \ldots, f(3,2,1))$ is a representative of the clone generated by f.

Idea

Only generate the “smallest” representatives with respect to the lexicographical order on tuples.

Reduce the number of semiprojections to check (step I)

Idea

If two functions differ only by permutation of variables, then they generate the same clone. We only need to take the "smaller" representative of the clone.
Many functions have the same permutation which give a "smaller" representative.

x_1	x_2	x_3	$f(x_1, x_2, x_3)$	$f(x_1, x_3, x_2)$
0	1	2	1	0
0	1	3	1	?
0	2	1	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Reduce the number of semiprojections to check (step II)

Let π be a permutation of $\{0, 1, \ldots, k-1\}$. Then $\sigma_\pi : P_k \to P_k$ defined by $\sigma_\pi(f) = \pi^{-1}(f(\pi(x_1), \ldots, \pi(x_n)))$ is called *inner automorphism* of P_k.

Idea (Inner Automorphisms)

For a minimal clone C generated by a semiprojection, it is easy to find all the other clones isomorphic to C w.r.t. an inner automorphism.
We only need to take the "smallest" representative of these clones.

Let $\pi(0) = 0, \pi(1) = 1, \pi(2) = 3, \pi(3) = 2$.

x_1	x_2	x_3	$f_a(\mathbf{x})$	$(\sigma_\pi f_0)(\mathbf{x})$	$(\sigma_\pi f_1)(\mathbf{x})$	$(\sigma_\pi f_2)(\mathbf{x})$	$(\sigma_\pi f_3)(\mathbf{x})$
0	1	2	2	0	1	3	2
0	1	3	a	?	?	?	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

This example reduces the number of candidates by $2 \cdot 4^{22}$ after only two places are known.

Reduce the number of semiprojections to check (step III)

Idea

We search for minimal clones. Thus every function (except the projections) contained therein generates it. Use compositions of partially defined functions to find "smaller" representatives in the clone.

x_1	x_2	x_3	$f(x_1, x_2, x_3)$	$f(x_1, x_2, f(x_1, x_2, x_3))$
0	1	2	1	$f(0, 1, 1) = 0$
0	1	3	2	$f(0, 1, 2) = 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Note: If the "smaller" representative does not generate a minimal clone, then the original one does not.

Results: 4-ary semiprojections

Theorem

There are 76 minimal clones generated by 4-ary semiprojections in 11 conjugacy classes.

smallest representative	S	A
112233002233001133001122	1	1
000033112211221122003333	3	3
000000111213222123333132	6	4
000000111111222233333322	3	6
000000002233222222333333	1	4
000000000333222222333333	6	12
000000000333000333333333	6	6
000000000000222222333333	3	12
000000000000222222222222	15	12
000000000000000000333333	15	12
000000000000000000000000	3	4

S: # of distinct (w.r.t. permutation of variables) 4-ary semiprojections

A: # of isomorphic clones (w.r.t. to inner automorphisms)

Interaction between semiprojections of different arities

Lemma

Let $f \in P_k^n$ be a semiprojection. Then $[f]$ is a minimal clone in P_k if and only if

- *there is no non-trivial n-ary semiprojection $g \in [f]^n$ with $[g]^n \subsetneq [f]^n$, and*
- *there is no non-trivial m-ary semiprojection $h \in [f]$ with $n < m \leq k$.*

Note: $3 = n < k = 4$ is the difficult part to follow.
Observation: there is a ternary semiprojection f with more than 50 non-trivial 4-ary functions.

Ternary semiprojections: Strategy I

There are 179 minimal representative (w.r.t. permutation of variables and inner automorphisms) which fulfill the first condition, i.e. semiprojections $f \in P_4^3$ such that no semiprojection $g \in P_4^3$ with $[g]^3 \subsetneq [f]^3$ exists.

Idea

There are the following two possibilities:

- *There is a 4-ary semiprojection in $[f]$. Then construct it, and thus f does not create a minimal clone.*
- *There is no 4-ary semiprojection in $[f]$. Then find for each minimal 4-ary semiprojection g, a relation $\varrho_{g,f}$ with $f \in \operatorname{Pol} \varrho_{g,f}$ and $g \notin \operatorname{Pol} \varrho_{g,f}$. Thus $[f]$ is a minimal clone.*

Ternary semiprojections: Strategy II

For 176 of 179 this worked by brute-force for the constructions and using only binary relations.

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$f(x_1, f(x_2, x_3, x_4), x_2)$
$f(x_1, x_2, f(x_1, x_3, x_4))$
$f(x_1, x_2, f(x_2, x_3, x_4))$
$f(x_1, f(x_1, x_2, x_3), f(x_1, x_4, x_3))$
$f(x_1, f(x_2, x_1, x_3), f(x_4, x_2, x_1))$
$f(x_1, x_2, f(f(x_2, x_1, x_3), x_4, x_2))$
$f(x_1, x_2, f(f(x_2, x_3, x_1), x_2, x_4))$
$f(x_1, x_2, f(x_1, x_3, f(x_3, x_2, x_4)))$
$f(x_1, f(x_2, x_1, x_3), f(f(x_4, x_1, x_3), x_2, x_3))$
$f(x_1, f(x_1, x_2, x_3), f(f(x_2, x_3, x_4), x_1, f(x_4, x_1, x_3)))$
$f(f(x_1, x_2, x_3), f(f(x_2, x_1, x_4), f(x_4, x_1, x_3), x_3), x_3)$
$f(x_1, x_3, f(f(x_1, x_3, f(x_4, x_2, x_3)), x_2, f(f(x_1, x_2, x_4), x_3, x_2)))$

Results: Ternary semiprojections

Theorem

There are 489 *minimal clones generated by ternary semiprojections in* 35 *conjugacy classes.*

smallest representative	S	A
000033112211221122003333	2	3
000002131111220222333331	2	6
000002111311220222133333	2	6
000000112131221232331323	2	4
000000111131221222333323	2	8
000000111111323222232333	1	6
000000111111223222233333	1	6
000000111111222233333322	2	6
000000111111222223333323	2	12
000000111010222222333333	1	12
000000111010222020333333	2	12
000000012111222222303323	1	6
000000010313222222333333	2	12
000000010313020323333333	2	6
000000010111232232333232	2	24
000000010111222323333333	2	12
000000010111222323323323	2	24
000000010111222222333232	2	24

smallest representative	S	A
000000010111222222333131	2	24
000000010111222222333333	2	24
000000010111222222323323	2	12
000000010111222222303303	2	12
000000010111222222303101	2	24
000000010010323323333333	2	12
000000010010222222333333	2	24
000000010010222222303000	2	12
000000010010020020333333	2	12
000000002131222222333333	1	12
000000002121222222333333	1	24
000000000101222323323323	2	24
000000000101222222333232	2	24
000000000101222222333333	2	12
000000000101222222323323	2	24
000000000101222222323222	2	12
000000000101020222030333	2	12

Observation: In a minimal clone generated by a ternary semiprojection in P_4 there are at most 2 distinct semiprojections.

S: # of distinct (w.r.t. permutation of variables) 3-ary semiprojections; A: # of isomorphic clones (w.r.t. to inner automorphisms)

Conclusion

Theorem

There are 5242 minimal clones on a four-element set.

	type	count	
(I)	unary functions	53	(Pöschel Kaluzhnin,1979)
(II)	binary idempotent functions	2182	(Szczepara, 1995)
(III)	majority functions	2441	(Waldhauser, 2008)
(IV)	minority functions	1	
(V)	semiprojections	565	
	sum	5242	

Table: The number of minimal clones on **4**

Conclusion

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Thank you for your attention.

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