Algebraic closure of some generalized convex sets

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AFFINE SPACES AND CONVEX SETS
1. Real affine spaces

Given a vector space (or a module) A over a subfield (or a subring) R of $\mathbb{R}$:

An **affine space** A **over** R (or **affine** R**-space**) is the algebra

$$\Big(A, \sum_{i=1}^{n} x_i r_i \,\Big|\, \sum_{i=1}^{n} r_i = 1\Big).$$

This algebra is equivalent to

$$(A, P, \underline{R}),$$

where

$$\underline{R} = \{\underline{f} \mid f \in R\}$$

and

$$xy\underline{f} := x(1-f) + yf =: \underline{f}(x,y),$$

and P is the Mal'cev operation

$$xyzP := x - y + z =: P(x,y,z).$$

The class $\underline{\underline{R}}$ of all affine R-spaces is a variety.

Abstractly, $\underline{\underline{R}}$ is defined as the class of idempotent entropic Mal'cev algebras $(A, P, \underline{R})$ with a ternary Mal'cev operation P and binary operations $\underline{r}$ for each $r \in R$, satisfying the identities:

$$\begin{aligned} &xy\underline{0} = x = yx\underline{1}, \\ &xy\underline{p}\ xy\underline{q}\ \underline{r} = xy\ \underline{pq\underline{r}}, \\ &xy\underline{p}\ xy\underline{q}\ xy\underline{r}\ P = xy\ \underline{pqrP} \end{aligned}$$

for all $p, q, r \in R$.

The variety $\underline{\underline{R}}$ also satisfies the **entropic** identities

$$xy\underline{p}\ zt\underline{p}\ \underline{q} = xz\underline{q}\ yt\underline{q}\ \underline{p}$$

for all $p, q \in R$ and the **cancellation laws**

$$(xy\underline{p} = xz\underline{p}) \to y = z$$

for all invertible $p \in R$.

2. Convex sets and barycentric algebras

Let F be a subfield of $\mathbb{R}$, $I^o(F) :=]0,1[\subset F$ and $I(F) := [0,1] \subset F$.

Convex subsets of affine F-spaces (or F-**convex sets**) are $\underline{I}^o(F)$-subreducts $(A, \underline{I}^o(F))$ of affine F-spaces.

The class $\mathcal{C}v(F)$ of F-convex sets generates the variety $\mathcal{B}(F)$ of F-**barycentric algebras**.

Theorem The class $\mathcal{C}v(F)$ and the quasivariety $\mathcal{C}(F)$ of cancellative F-barycentric algebras coincide. $\mathcal{C}v(F)$ is a minimal subquasivariety of the variety $\mathcal{B}(F)$.

3. Intervals of the F-line

The algebra $(F, \underline{F})$ is called an F-**line**, and intervals of the F-line are closed bounded intervals considered as $\underline{I}^o(F)$-algebras.

Proposition The following conditions are equivalent for any non-trivial subalgebra $(A, \underline{I}^o(F))$ of $(F, \underline{I}^o(F))$:

(a) $(A, \underline{I}^o(F))$ is a closed interval of $(F, \underline{I}^o(F))$;

(b) $(A, \underline{I}^o(F))$ is isomorphic to $(I(F), \underline{I}^o(F))$;

(c) $(A, \underline{I}^o(F))$ is generated by two (distinct) elements;

(d) $(A, \underline{I}^o(F))$ is a free algebra on two free generators in the quasivariety $\mathcal{C}(F)$ and in the variety $\mathcal{B}(F)$.

4. R-convex sets

Now assume that R is a principal ideal subdomain of $\mathbb{R}$ such that $\mathbb{Z} \subset R \subseteq \mathbb{R}$.
The algebra $(R, P, \underline{R})$ is called an R-**line**.

Note that not all intervals of the line $(R, P, \underline{R})$ are isomorphic to the unit interval $(I(R), \underline{I}^o(R))$, and not all are generated by its endpoints.

However $(I(R), \underline{I}^o(R))$ is generated by the endpoints and is free on two generators, in the quasivariety and the variety it generates.

Let $I^o(R) :=]0, 1[\subset R$.

Algebraic R-convex subsets of affine R-spaces are $I^o(R)$-subreducts $(A, \underline{I}^o(R))$ of faithful affine R-spaces.

Geometric R-convex sets of affine R-spaces R^n are the intersections of $\mathbb{R}$-convex subsets of $\mathbb{R}^n$ with the subspace R^n.

If R is a field, both concepts coincide.
If not, then the algebraic and geometric definitions of R-convex sets do not coincide.

Proposition The class $\mathcal{C}v(R)$ of $\underline{I}^o(R)$-subreducts of faithful affine R-spaces is a (minimal) quasivariety containing the class of geometric R-convex sets.

$\mathcal{C}v(R)$ **does not** coincide with the quasivariety of cancellative members of the variety generated by $\underline{I}^o(R)$-subreducts of affine R-spaces.

MODES

An algebra (A, Ω) is a **mode** if it is

- **idempotent**:

$$x...x\omega = x,$$

  for each n-ary $\omega \in \Omega$, and

- **entropic**:

$$(x_{11}...x_{1n}\omega)...(x_{m1}...x_{mn}\omega)\,\varphi$$

$$= (x_{11}...x_{m1}\varphi)...(x_{1n}...x_{mn}\varphi)\,\omega.$$

  for all $\omega, \varphi \in \Omega$.

Affine R-spaces, R-convex sets and their subreducts are modes.

ALGEBRAIC CLOSURES OF GEOMETRIC R-CONVEX SETS

From now on, R is a principal ideal subdomain of $\mathbb{R}$ such that $\mathbb{Z} \subset R \subseteq \mathbb{R}$, and
$I^o(R)$ contains an invertible element s.

All R-convex sets $(C, \underline{I}^o(R))$ are assumed to be geometric subsets of an affine R-space A isomorphic to $(R^k, P, \underline{R})$ for some $k = 1, 2, \ldots.$

For $(a, b) \in C \times C$,
$\langle a, b \rangle$ denotes the $\underline{I}^o(R)$-subalgebra generated by a and b, and
$\langle a, b \rangle^o := \langle a, b \rangle \setminus \{a, b\}$.

1. Algebraic s-closures

The pair (a, b) is called s-**eligible**, if for each $x \in \langle a, b\rangle^o$ there is a $y \in C$ with $b = xy\underline{s}$.
$E_s(C)$ denotes the set of s-eligible pairs of $(C, \underline{I}^o(R))$.

Lemma The set $E_s(C)$ forms a subalgebra of $(A \times A, \underline{I}^o(R))$.

Lemma Let $(a, b) \in C \times C$. Then (a, b) is an s-eligible pair of $(C, \underline{I}^o(R))$ if and only if $xb\underline{1/s} \in C$ for each $x \in \langle a, b\rangle^o$.

An R-convex subset $(C, \underline{I}^o(R))$ of an affine R-space A is called **algebraically** s-**closed** if for each s-eligible pair $(a, b) \in C \times C$, there is a $c \in C$ such that $b = ac\underline{s}$.

Proposition An R-convex subset $(C, \underline{I}^o(R))$ of an affine R-space A is algebraically s-closed if and only if $ab\underline{1/s} \in C$ for each s-eligible pair $(a, b) \in C \times C$.

Let

$$\overline{C}_s := \{ab\underline{1/s} \mid (a, b) \in E_s(C)\}.$$

The set $\overline{C}_s$ is called the **algebraic s-closure** of $(C, \underline{I}^o(R))$.

Lemma The s-closure $\overline{C}_s$ of an R-convex subset $(C, \underline{I}^o(R))$ of an affine R-space A is a subalgebra of $(A, \underline{I}^o(R))$.

Lemma Let s and t be any two invertible elements of $I^o(R)$. Then $\overline{C}_s$ and $\overline{C}_t$ coincide.

2. Algebraic closures

The s-closure $\overline{C}_s$ of C will be called the **algebraic closure** or simply the **closure** of C, and will be denoted by $\overline{C}$.

Proposition Let C be a k-dimensional geometric convex subset of the affine R-space R^k. Then its closure $\overline{C}$ is also a geometric k-dimensional convex subset of R^k, and it coincides with the convex hull $\mathtt{conv}_R(\overline{C})$ of $\overline{C}$.

Proposition The following hold for the closures $\overline{B}$ and $\overline{C}$ of R-convex subsets $(C, \underline{I}^o(R))$ and $(B, \underline{I}^o(R))$ of an affine R-space R^k.

(a) $C \leq \overline{C}$;

(b) If $(B, \underline{I}^o(R)) \leq (C, \underline{I}^o(R))$,
then $(\overline{B}, \underline{I}^o(R)) \leq (\overline{C}, \underline{I}^o(R))$;

(c) $\overline{\overline{C}} = \overline{C}$.

ALGEBRAIC AND OTHER CLOSURES

Consider an affine R-space $(A, P, \underline{R})$. Define the following relation $\sim_s$ on the set $A \times A$:

$$(a_1, b_1) \sim_s (a_2, b_2) \text{ iff } a_1 b_2 \underline{s} = a_1 a_2 \underline{s}\, b_1 \underline{s}.$$

Lemma (a) The relation $\sim_s$ is a congruence relation of the affine R-space $(A \times A, P, \underline{R})$.
(b) The mapping

$$\varphi : A \to (A \times A)^{\sim_s} \text{ ; } a \mapsto (a, a)^{\sim_s}$$

is an embedding of affine R-spaces.
(c) The relation $\sim_s$ is a congruence relation of $\underline{I}^o(R)$-subreducts of $(A \times A, P, \underline{R})$, in particular of each R-convex set $(C \times C, \underline{I}^o(R))$.

Lemma Let $(A, \underline{I}^o(R))$ be the $\underline{I}^o(R)$-reduct of an affine R-space $(A, P, \underline{R})$. Then

$$(E_s(A), \underline{I}^o(R))^{\sim_s} \cong (A, \underline{I}^o(R))$$

.

1. Algebraic closures and aiming congruences

The congruence $\sim_s$ of $(C \times C, \underline{I}^o(R))$ is called the **aiming congruence**.

Proposition Let $(C, \underline{I}^o(R))$ be an R-convex subset of an affine R-space $(A, P, \underline{R})$. Then

$$(\overline{C}_s, \underline{I}^o(R)) \cong (E_s(C), \underline{I}^o(R))^{\sim_s}.$$

Corollary The following conditions are equivalent for a k-dimensional geometric R-convex subset C of the affine R-space R^k, where $k = 1, 2, \ldots$, and an invertible element $s \in I^o(R)$:

(a) $(C, \underline{I}^o(R))$ is algebraically closed,
(b) $(C, \underline{I}^o(R)) \cong (\overline{C}, \underline{I}^o(R))$,
(c) $(C, \underline{I}^o(R)) \cong (E_s(C), \underline{I}^o(R))^{\sim_s}$.

2. Algebraic and topological closures

We consider the usual Euclidean topology on $\mathbb{R}^k$, and R^k as a topological subspace of $\mathbb{R}^k$. Its closed (open) sets are simply closed (open) subsets of $\mathbb{R}^k$ intersected with R^k.

For a geometric convex subset C of R^k, let C_R^{tc} be its topological closure in R^k, and $C_{\mathbb{R}}^{tc}$ its topological closure in $\mathbb{R}^k$.

Theorem Let $(C, \underline{I}^o(R))$ be a k-dimensional geometric convex subset of an affine R-space $(R^k, P, \underline{R})$. Then the algebraic closure $\overline{C}$ of C and the topological closure C_R^{tc} of C in R^k coincide:

$$\overline{C} = C_R^{tc}.$$