Representation of distributive spatial lattices by congruence lattices of groupoids

Alexander Popovich

Szeged, June 2012

Let L be a complete lattice and $a \in L$.

- a is *compact* if $a \leqslant \bigvee X$, for any $X \subseteq L$, implies that $a \leqslant \bigvee X'$ for some finite $X' \subseteq X$.

Definitions

Let L be a complete lattice and $a \in L$.

- a is *compact* if $a \leqslant \bigvee X$, for any $X \subseteq L$, implies that $a \leqslant \bigvee X'$ for some finite $X' \subseteq X$.
- L is *algebraic* if every it's element is a join of compact elements.

Definitions

Let L be a complete lattice and $a \in L$.

- a is *compact* if $a \leqslant \bigvee X$, for any $X \subseteq L$, implies that $a \leqslant \bigvee X'$ for some finite $X' \subseteq X$.
- L is *algebraic* if every it's element is a join of compact elements.
- a is *completely join-irreducible* if $a = \bigvee X$, for any $X \subseteq L$, implies that $a \in X$.

Definitions

Let L be a complete lattice and $a \in L$.

- a is *compact* if $a \leqslant \bigvee X$, for any $X \subseteq L$, implies that $a \leqslant \bigvee X'$ for some finite $X' \subseteq X$.
- L is *algebraic* if every it's element is a join of compact elements.
- a is *completely join-irreducible* if $a = \bigvee X$, for any $X \subseteq L$, implies that $a \in X$.
- L is *spatial* if every it's element is a join of completely join-irreducible elements.

Definitions

Let L be a complete lattice and $a \in L$.

- a is *compact* if $a \leqslant \bigvee X$, for any $X \subseteq L$, implies that $a \leqslant \bigvee X'$ for some finite $X' \subseteq X$.
- L is *algebraic* if every it's element is a join of compact elements.
- a is *completely join-irreducible* if $a = \bigvee X$, for any $X \subseteq L$, implies that $a \in X$.
- L is *spatial* if every it's element is a join of completely join-irreducible elements.
- Every distributive spatial lattice is algebraic.

Representation of algebraic lattices

- G. Grätzer, E.T. Schmidt (1963) *Every algebraic lattice can be represented as the congruence lattice of some algebra.*

Representation of algebraic lattices

- G. Grätzer, E.T. Schmidt (1963) *Every algebraic lattice can be represented as the congruence lattice of some algebra.*
- R. Freese, W.A. Lampe, W. Taylor (1979) *There exist a modular lattice which is not represented as the congruence lattice of any algebra with finite type.*

Representation of algebraic lattices

- G. Grätzer, E.T. Schmidt (1963) *Every algebraic lattice can be represented as the congruence lattice of some algebra.*
- R. Freese, W.A. Lampe, W. Taylor (1979) *There exist a modular lattice which is not represented as the congruence lattice of any algebra with finite type.*
- W.A. Lampe (1982) *Every algebraic lattice with compact unit is isomorphic to the congruence lattice of some groupoid.*

- **Open problem**. Is every distributive algebraic lattice isomorphic to the congruence lattice of some groupoid (or, moreover, semigroup)?

Representation of distributive algebraic lattices

- **Open problem**. Is every distributive algebraic lattice isomorphic to the congruence lattice of some groupoid (or, moreover, semigroup)?
- E.T. Schmidt (1981) *Every distributive algebraic lattice whose compact elements form a lattice is isomorphic to the congruence lattice of some lattice with 0.*

- **Open problem**. Is every distributive algebraic lattice isomorphic to the congruence lattice of some groupoid (or, moreover, semigroup)?
- E.T. Schmidt (1981) *Every distributive algebraic lattice whose compact elements form a lattice is isomorphic to the congruence lattice of some lattice with 0.*
- P. Ružička, J. Tůma, F. Wehrung (2005) *Every distributive algebraic lattice whose compact elements have cardinality* $\leqslant \aleph_1$ *is isomorphic to the congruence lattice of some group.*

Ash's result

- *An ideal* of a semigroup S is a set $I \subseteq S$, such that $IS \subseteq I$ and $SI \subseteq I$.

Ash's result

- *An ideal* of a semigroup S is a set $I \subseteq S$, such that $IS \subseteq I$ and $SI \subseteq I$.
- *A principal ideal* is an ideal generated by one element.

Ash's result

- *An ideal* of a semigroup S is a set $I \subseteq S$, such that $IS \subseteq I$ and $SI \subseteq I$.
- *A principal ideal* is an ideal generated by one element.
- Joins and meets of ideals are set-theretic.

Ash's result

- *An ideal* of a semigroup S is a set $I \subseteq S$, such that $IS \subseteq I$ and $SI \subseteq I$.
- *A principal ideal* is an ideal generated by one element.
- Joins and meets of ideals are set-theretic.
- All principal ideals are completely join-irredusible.

- *An ideal* of a semigroup S is a set $I \subseteq S$, such that $IS \subseteq I$ and $SI \subseteq I$.
- *A principal ideal* is an ideal generated by one element.
- Joins and meets of ideals are set-theretic.
- All principal ideals are completely join-irredusible.
- Every ideal of a semigroup S is the join of it's principal ideals, so the ideal lattice $\mathrm{Id}\, S$ is distributive and spatial.

Ash's result

- *An ideal* of a semigroup S is a set $I \subseteq S$, such that $IS \subseteq I$ and $SI \subseteq I$.
- *A principal ideal* is an ideal generated by one element.
- Joins and meets of ideals are set-theretic.
- All principal ideals are completely join-irredusible.
- Every ideal of a semigroup S is the join of it's principal ideals, so the ideal lattice $\mathrm{Id}\, S$ is distributive and spatial.
- C.J. Ash (1980) *Every disributive spatial lattice is isomorphic to the ideal lattice of some semigroup.*

Theorem. Every distributive spatial lattice is isomorphic to the congruence lattice of some groupoid G, such that G satisfies the identities $xy = yx$ and $x^2 = 0$ and $\mathrm{Id}\, G \cong \mathrm{Con}\, G$.

- Let G be a groupoid with 0 and P is a $(\vee, 0)$-semilattice.
  A map $\rho : G \to P$ is an *ideal function* if
  1. $\rho(0) = 0$
  2. $\rho(xy) \leqslant \rho(x)$ and $\rho(xy) \leqslant \rho(y)$

Ideal function

- Let G be a groupoid with 0 and P is a $(\vee, 0)$-semilattice.
  A map $\rho : G \to P$ is an *ideal function* if
  1. $\rho(0) = 0$
  2. $\rho(xy) \leqslant \rho(x)$ and $\rho(xy) \leqslant \rho(y)$
- Let I be an ideal in P.
  Then a set $\rho^{-1}(I) = \{x \in G : \rho(x) \in I\}$ is an ideal of G.
  So, $\rho^{-1} : \mathrm{Id}\, P \to \mathrm{Id}\, G$

Proposition. Let G be a groupoid with 0, P $\vee$-semilattice with 0 and $\rho : G \to P$ an ideal function.
Let $a, b \in G$ satisfy $\rho(a) \geqslant \rho(b)$.
Then there exists a groupoid $\tilde{G}$ with 0 and an ideal function $\tilde{\rho} : \tilde{G} \to P$ such that
1) G is a subgroupoid of $\tilde{G}$;
2) $\tilde{\rho}|_G = \rho$;
3) there exists an element $u \in \tilde{G}$ such that $au = b$ and $xu = 0$ for $x \in \tilde{G} \backslash \{a\}$;
4) if G satisfies $xy = yx$ and $x^2 = 0$, then so $\tilde{G}$.

Corollary. Let G be a groupoid with 0, P $\vee$-semilattice with 0 and $\rho : G \to P$ an ideal function.
Then there exists a groupoid $\tilde{G}$ with 0 and an ideal function $\tilde{\rho} : \tilde{G} \to P$ such that
1) G is a subgroupoid of $\tilde{G}$;
2) $\tilde{\rho}|_G = \rho$;
3) for every a, b with $\rho(a) \geqslant \rho(b)$ there exists an element $u \in \tilde{G}$ such that $au = b$ and $xu = 0$ for $x \in \tilde{G} \backslash \{a\}$;
4) if G satisfies $xy = yx$ and $x^2 = 0$, then so $\tilde{G}$.

First step

- Let L be a distributive spatial lattice. $\mathrm{Ji}\, L$ is the set of join-irredusible elements.

First step

- Let L be a distributive spatial lattice. $\mathrm{Ji}\, L$ is the set of join-irredusible elements.
- A subset $A \subseteq \mathrm{Ji}\, L$ is called a down-set if $x \in A$ and $y \leqslant x$ imply that $y \in A$. $\mathrm{Down}\,\mathrm{Ji}\, L$ is the set of all down-sets of $\mathrm{Ji}\, L$

First step

- Let L be a distributive spatial lattice. $\mathrm{Ji}\, L$ is the set of join-irredusible elements.
- A subset $A \subseteq \mathrm{Ji}\, L$ is called a down-set if $x \in A$ and $y \leqslant x$ imply that $y \in A$. $\mathrm{Down}\, \mathrm{Ji}\, L$ is the set of all down-sets of $\mathrm{Ji}\, L$
- There exist an isomorphism $\phi : L \to \mathrm{Down}\, \mathrm{Ji}\, L$

First step

- Let L be a distributive spatial lattice. $\mathrm{Ji}\,L$ is the set of join-irredusible elements.
- A subset $A \subseteq \mathrm{Ji}\,L$ is called a down-set if $x \in A$ and $y \leqslant x$ imply that $y \in A$. $\mathrm{Down}\,\mathrm{Ji}\,L$ is the set of all down-sets of $\mathrm{Ji}\,L$
- There exist an isomorphism $\phi : L \to \mathrm{Down}\,\mathrm{Ji}\,L$
- Set $G = \mathrm{Ji}\,L \cup \{0\}$ and form a groupoid with zero multiplication ($xy = 0$). Set $\rho : G \to \mathrm{Comp}\,L$ by the rule $\rho(x) = \phi(x)$ for $x \in G$ and $\rho(0) = 0$.

Then there exists a groupoid $\tilde{G}$ with zero 0, satisfying $xy = yx$ and $x^2 = 0$, and an ideal function $\tilde{\rho} : \tilde{G} \to \mathrm{Comp}\, L$ such that G is a subgroupoid of $\tilde{G}$, $\tilde{\rho}|G = \rho$ and
1) for every $x \in \tilde{G}$ there exists $u_x \in \tilde{G}$ such that $xu_x = x$ and $yu_x = 0$ for $y \in \tilde{G}\backslash\{x\}$;
2) for every $x, y \in \tilde{G}$ if $\tilde{\rho}(x) \geqslant \tilde{\rho}(y)$, then there exists $v \in \tilde{G}$ such that $xv = y$.

Final step

- Let $\Theta \in \mathrm{Con}\,\tilde{G}$ and $(a, b) \in \Theta$.

Final step

- Let $\Theta \in \mathrm{Con}\, \tilde{G}$ and $(a, b) \in \Theta$.
- Then $(a, 0) = (au_a, bu_a) \in \Theta$ and $(b, 0) = (au_b, bu_b) \in \Theta$. So $\Theta = I \times I$ for some ideal I.

Final step

- Let $\Theta \in \mathrm{Con}\, \tilde{G}$ and $(a, b) \in \Theta$.
- Then $(a, 0) = (au_a, bu_a) \in \Theta$ and $(b, 0) = (au_b, bu_b) \in \Theta$. So $\Theta = I \times I$ for some ideal I.
- We got $\mathrm{Con}\, \tilde{G} \cong \mathrm{Id}\, \tilde{G}$.

Final step

- Let $\Theta \in \mathrm{Con}\,\tilde{G}$ and $(a, b) \in \Theta$.
- Then $(a, 0) = (au_a, bu_a) \in \Theta$ and $(b, 0) = (au_b, bu_b) \in \Theta$. So $\Theta = I \times I$ for some ideal I.
- We got $\mathrm{Con}\,\tilde{G} \cong \mathrm{Id}\,\tilde{G}$.
- Let $\mathrm{Id}^1\,\tilde{G}$ be the set of principal ideals of $\tilde{G}$. Set $\psi : \mathrm{Ji}\,L \to \mathrm{Id}^1\,\tilde{G}$ by the rule $\psi(p) = \{x \in \tilde{G} | \rho(x) \leqslant p\}$.

Final step

- Let $\Theta \in \mathrm{Con}\,\tilde{G}$ and $(a, b) \in \Theta$.
- Then $(a, 0) = (au_a, bu_a) \in \Theta$ and $(b, 0) = (au_b, bu_b) \in \Theta$. So $\Theta = I \times I$ for some ideal I.
- We got $\mathrm{Con}\,\tilde{G} \cong \mathrm{Id}\,\tilde{G}$.
- Let $\mathrm{Id}^1\,\tilde{G}$ be the set of principal ideals of $\tilde{G}$. Set $\psi : \mathrm{Ji}\,L \to \mathrm{Id}^1\,\tilde{G}$ by the rule $\psi(p) = \{x \in \tilde{G} | \rho(x) \leqslant p\}$.
- ψ is an isomorphism.

Final step

- Let $\Theta \in \operatorname{Con} \tilde{G}$ and $(a, b) \in \Theta$.
- Then $(a, 0) = (au_a, bu_a) \in \Theta$ and $(b, 0) = (au_b, bu_b) \in \Theta$. So $\Theta = I \times I$ for some ideal I.
- We got $\operatorname{Con} \tilde{G} \cong \operatorname{Id} \tilde{G}$.
- Let $\operatorname{Id}^1 \tilde{G}$ be the set of principal ideals of $\tilde{G}$. Set $\psi : \operatorname{Ji} L \to \operatorname{Id}^1 \tilde{G}$ by the rule $\psi(p) = \{x \in \tilde{G} | \rho(x) \leqslant p\}$.
- ψ is an isomorphism.
- Then $L \cong \operatorname{Down} \operatorname{Ji} L \cong \operatorname{Down} \operatorname{Id}^1 \tilde{G} \cong \operatorname{Id} \tilde{G}$.

Thank you for attention.