A new operation on finite partially ordered sets inherited from the random poset

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Joint with Csaba Szabó

Graphs

Binary relation

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1. aEb

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2. aNb

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3. (or $a = b$)

Operations on graphs

Complementation

$E \leftrightarrow N$

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switch (J. J. Seidel)

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3. identical otherwise

Why are they special?

Countable homogeneous structures

Definition

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Example

$\Gamma = (V; E)$: random graph

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- $\langle \mathrm{Aut}(\Gamma), -, \mathrm{sw} \rangle$
- $\mathrm{Sym}(\Gamma)$

Fast algorithms

Switching to a

1. triangle-free graph. (R. B. Hayward, 1996; and J. Hage, T. Harju, E. Welzl, 2002)
2. planar graph. (A. Ehrenfeucht, J. Hage, T. Harju, G. Rozenberg, 2000; J. Kratochvil, 2003)
3. Eulerian graph. (J. Hage, T. Harju, E. Welzl, 2002)
4. bipartite graph. (J. Hage, T. Harju, E. Welzl, 2002)
5. claw-free graph. (E. Jelinkova, J. Kratochvil, 2008)

Slow algorithm

Switching to a regular graph. (Kratochvil, 2003)

Cameron's theorem

Total orders

$(\mathbb{Q}, <)$

Theorem (1976)

Let G be a closed group containing $\mathrm{Aut}(\mathbb{Q}, <)$. Then G is one of the following.

- $\mathrm{Aut}(\mathbb{Q}, <)$
- $\langle \mathrm{Aut}(\mathbb{Q}, <), \updownarrow \rangle$
- $\langle \mathrm{Aut}(\mathbb{Q}, <), \mathrm{cycl} \rangle$
- $\langle \mathrm{Aut}(\mathbb{Q}, <), \updownarrow, \mathrm{cycl} \rangle$
- $\mathrm{Sym}(\mathbb{Q})$

Partial orders

Binary relation

1. $a < b$
2. $a > b$
3. $a \perp b$
4. (or $a = b$)

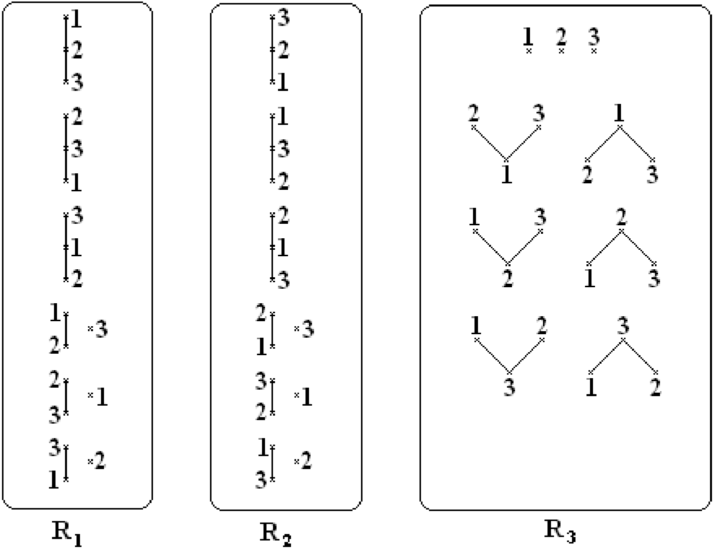
Rotations

Definition

A bijective map $f : A \to B$ is a poset rotation if there exists a partition $A = X \cup Y \cup Z$ such that

1. X is downward closed, Z is upward closed, and $X < Z$,
2. $f|_X$, $f|_Y$ and $f|_Z$ are isomorphisms,
3. if $x \perp y$ then $f(x) > f(y)$,
4. if $x < y$ then $f(x) \perp f(y)$,
5. if $y \perp z$ then $f(y) > f(z)$,
6. if $y < z$ then $f(y) \perp f(z)$,
7. $f(X) > f(Z)$.

3-orbits



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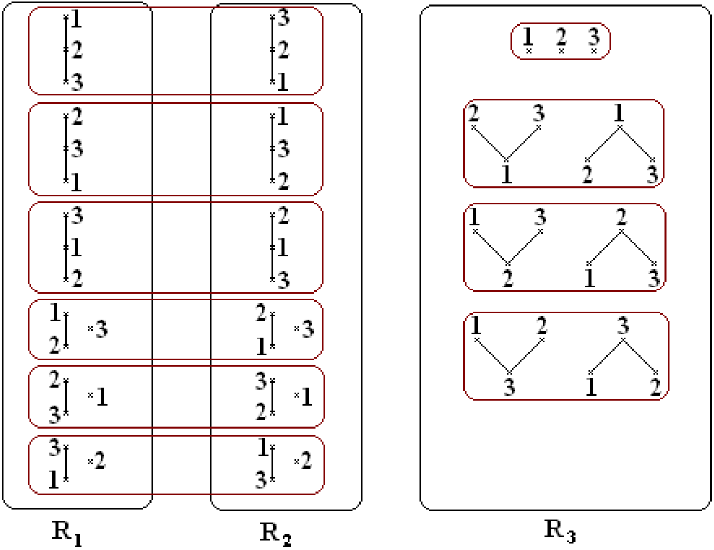
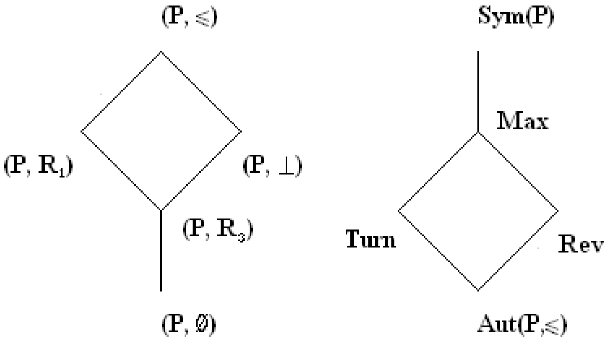


Theroem (P. P. Pach, M. Pinsker, G. Pluhár, A. P., Cs. Szabó)

$\mathcal{P}$ has 5 reducts up to first-order interdefinability:



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1. a method developed by M. Bodirsky and M. Pinsker
2. structural Ramsey theory (work of Nešetřil, Fouché, Sokic and others)

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Let $N \subseteq A$ be the linear sum of (at most) two antichains. Then there is a unique way to alter the relation on A so that R_1, R_2, R_3 are preserved and N becomes the set of maximal elements.

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Let $N \subseteq A$ be the linear sum of (at most) two antichains. Then there exists a rotation f on A such that $f(N)$ is the set of maximal elements in $f(A)$.

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Remark

$\frac{2^n}{n^2}$ can be obtained.

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- Is there a fast algorithm that decides whether a given finite poset is rotation equivalent with a poset having a "nice" property?
- Given an n-element set X with three ternary relations S_1, S_2, S_3. Is it in P to decide whether there exists a partial order $\leq$ on X such that $R_i = S_i$?