Congruence FD-maximal varieties

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Congruence lattices

Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

Or, at least,

describe the finite members of Con $\mathcal{K}$.

Necessary condition

In the sequel: $\mathcal{V}\dots$ a finitely generated CD variety;
SI$(\mathcal{V})\dots$ the family of subdirectly irreducible members;
$M(L)\dots$ completely $\wedge$-irreducible elements of a lattice L.

Lemma

Let $L \in$Con$\mathcal{V}$. Then for every $x \in M(L)$, the lattice $\uparrow x$ is isomorphic to $\mathrm{Con}\, T$ for some $T \in SI(\mathcal{V})$.

On the finite level (for finite L), the necessary condition is sometimes also sufficient. In such a case we say that $\mathcal{V}$ is *congruence FD-maximal*. Formally,$\mathcal{V}$ is congruence FD-maximal, if for every finite distributive lattice L the following two conditions are equivalent:

(i) $L \in \operatorname{Con} \mathcal{V}$;

(ii) for every $x \in M(L)$, the lattice $\uparrow x$ is isomorphic to $\operatorname{Con} T$ for some $T \in SI(\mathcal{V})$.

Congruence FD-maximal algebras

Let A be finite, generating a CD variety. We say that A is congruence FD-maximal, if for every finite distributive lattice L the following two conditions are equivalent:

(i) $L \cong \operatorname{Con} B$ for some $B \in P_s H(A)$;

(ii) for every $x \in M(L)$, the lattice $\uparrow x$ is isomorphic to $\operatorname{Con} T$ for some $T \in H(A)$.

Connection

Conjecture: the following condition are equivalent:

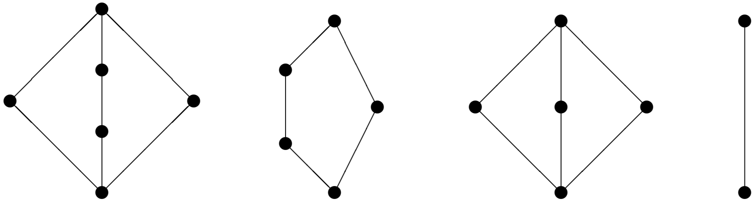
(i) $\mathcal{V}$ is congruence FD-maximal;

(ii) for every $T \in SI(\mathcal{V})$ there exists $A_T \in SI(\mathcal{V})$ such that

- A_T is congruence maximal;
- $\mathrm{Con}\, A_T \cong \mathrm{Con}\, T$;
- if $\mathrm{Con}(A_T/\tau) \cong \mathrm{Con}(A_S/\sigma)$ for some $S, T \in SI(\mathcal{V})$ and $\tau \in \mathrm{Con}\, A_T$, $\sigma \in \mathrm{Con}\, A_S$, then $(A_T/\tau) \cong (A_S/\sigma)$.

Example

Let $\mathcal{V} = HSP(K)$, where K is the lattice



K N_5 M_3 C_2

Subdirectly irreducible members are C_2, M_3, N_5 and K. Now, M_3 is a quotient of K, C_2 is a quotient of N_5, and $\mathrm{Con}\, C_2 \cong \mathrm{Con}\, M_3$, while C_2 and M_3 are not isomorphic.

Easy case

Theorem

If A is finite, generates a CD variety, and $\mathrm{Con}\, A$ is a chain, then A is congruence FD-maximal.

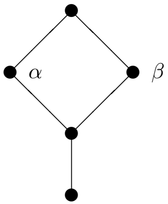
Theorem

Let $\mathcal{V}$ be a finitely generated congruence-distributive variety with the property that $\mathrm{Con}\, C$ is a finite chain for every $C \in \mathrm{SI}(\mathcal{V})$. Then $\mathcal{V}$ is congruence FD-maximal

Examples: distributive lattices, Stone algebras ...

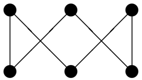
The simplest of the difficult cases

Let $\mathrm{Con}\, A$ be isomorphic to the following lattice V:



Non-congruence-maximal example

If the two nontrivial subdirectly irreducible quotients of A are not isomorphic, then A is not congruence FD-maximal. (The free distributive lattice with 3 generators does not belong to Con$P_sH(A)$. In this case, $M(L)$ is



Let E be a subset of $B \times B$ for some set B. Let X be a set and let $\mathcal{F}$ be a set of functions $X \to B$. We say that $\mathcal{F}$ is *E-compatible* if $\{f(x), g(x)) \mid x \in X\} = E$ or $\{(g(x), f(x)) \mid x \in X\} = E$ for every $f, g \in \mathcal{F}$, $f \neq g$.

Lemma

Suppose that $E \subseteq B \times B$ contains a pair (a, b) with $a \neq b$. Then the following condition are equivalent.

(i) *There exist arbitrarily large finite E-compatible sets of functions.*

(ii) *For every $(a, b) \in A$ there are $x, y, z \in B$ such that $(x, x), (y, y), (z, z), (x, y),$ $(x, z), (y, z), (x, a), (x, b), (a, y), (y, b), (a, z), (b, z) \in A.$*

Characterization theorem

Theorem

A is congruence-maximal if and only if the quotients A/α and A/β are isomorphic to the same algebra B and there exist surjective homomorphisms $h_0, h_1 : \ A \to B$ such that

(i) $\mathrm{Ker}(h_0) = \alpha$, $\mathrm{Ker}(h_1) = \beta$;

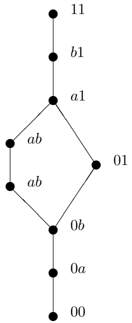
(ii) *there are arbitrarily large E-compatible sets of functions for $E = \{(h_0(x), h_1(x)) \mid x \in A\} \subseteq B \times B$.*

Positive example

For $A = N_5$ we have $B = \{0, 1\}$, $E = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ so almost every family of functions is E-compatible and N_5 is congruence FD-maximal.

Negative example

Consider the following lattice A with two additional unary operations.



$f(00) = 00, f(0a) = 0b, f(0b) = f(01) = 01$

$f(ab) = f(a1) = b1, f(b1) = f(11) = 11$

$g(11) = 11, g(b1) = a1, g(a1) = g(01) = 01$

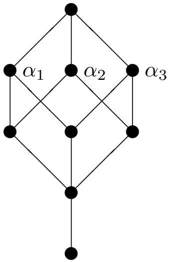
$g(ab) = g(0b) = 0a, g(0a) = g(00) = 00$

Negative example 2

We have $B = \{0, 1, a, b\}$,
$E = \{(0,0), (0,a), (0,b), (a,b), (0,1), (a,1), (b,1), (1,1)\}$ (the labels on the elements of A), and the pair (a,b) violates our condition. Thus, A is not congruence-maximal.

A generalization

Let $\mathrm{Con}\, A$ be an ordinal sum $\mathbf{1} \oplus P_n$, where P_n is the n-dimensional cube, with coatoms denoted by $\alpha_1, \ldots, \alpha_n$. For $n = 3$:



Compatible families generalized

Let E be a subset of B^m for some set B and $m > 1$. For a permutation π on $\{1, \ldots, m\}$ denote

$$E^\pi = \{(\pi(x_1), \ldots, \pi(x_m)) \mid (x_1, \ldots, x_m) \in E\}.$$

Let X be a set and let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a set of functions $X \to B$. We say that $\mathcal{F}$ is E-*compatible* if for every $i_1 < i_2 < \cdots < i_m \leq n$ there exists a permutation π such that $\{(f_{i_1}(x), \ldots f_{i_m}(x)) \mid x \in X\} = E^\pi$.

Compatible families generalized 2

Lemma

Suppose that $E \subseteq B \times B$ contains a nondiagonal m-tuple. Then the following condition are equivalent.

(i) *There exist arbitrarily large finite E-compatible sets of functions.*

(ii) *There exists π such that for every $(x_1, x_3, \dots, x_{2m-1}) \in E^\pi$ there exist $x_0, x_2, \dots, x_{2m} \in B$ such that*

$$(x_{i_1}, \dots, x_{i_m}) \in E^\pi$$

whenever $i_1 \leq i_2 \leq \cdots \leq i_m$ and odd indexes do not repeat.

Characterization theorem generalized

Theorem

A is congruence-maximal if and only if the quotients A/α_i ($i = 1, \dots, n$) are all isomorphic to the same algebra B and there exist surjective homomorphisms $h_1, \dots, h_m : A \to B$ such that

(i) $\mathrm{Ker}(h_i) = \alpha_i$ *for every i;*

(ii) *there are arbitrarily large E-compatible sets of functions for $E = \{(h_1(x), \dots, h_m(x)) \mid x \in A\} \subseteq B^m$.*