On the unimodality of rank numbers in face lattices of certain polytopes

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Unimodality conjecture

$$(1, 10, 45, 120, 205, 222, 140, 40, 1)$$

$$v_0 \leq v_1 \leq \ldots \leq v_k \geq \ldots \geq v_{d-1} \geq v_d .$$

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- Björner (1980)

Convex polytopes and their faces

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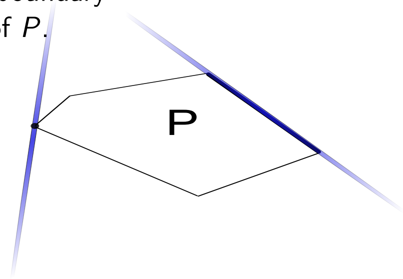
Let the polytope P be contained in a halfspace. The intersection of the boundary of this halfspace with P is a *face* of P.



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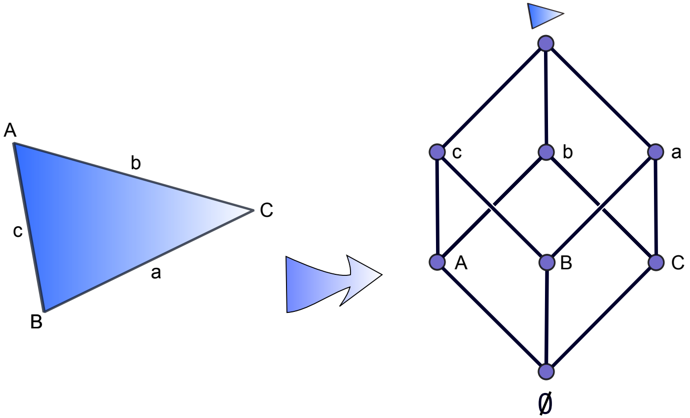

Rank numbers of face lattices

Face vector

For a finite graded lattice of height d consider the vector $(r_0, \ldots, r_d)$, where r_i is the number of elements of rank i. For face lattices of polytopes this vector is called face vector.

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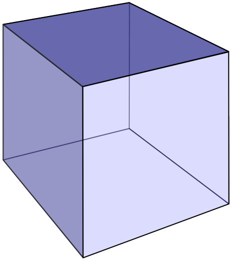
For example the face vector of the 3-cube is

$$(1, 8, 12, 6, 1)$$



Neighbourly and simplicial polytopes

If for any $k \leq d/2)$ atoms of the face lattice of polytope P, the principal ideal under the supremum of these k elements forms a boolean lattice, then P is called a *neighbourly polytope*. In 3 dimension every polytope is neighbourly.

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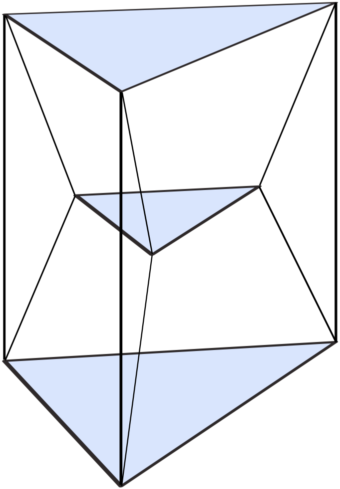
Polytope P is said to be *simplicial*, if for every coatom f of the face lattice of P, the interval $[0, f]$ is a boolean lattice.

Duality

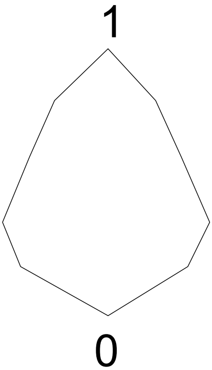
Polytope P^* is the dual polytope of P, if the face lattice of P^* is the dual lattice of the face lattice of P. The cube and the octahedron are dual to each other. The dual of a simplicial polytope is said to be *simple polytope*. The cube is simple and the octahedron is simplicial.

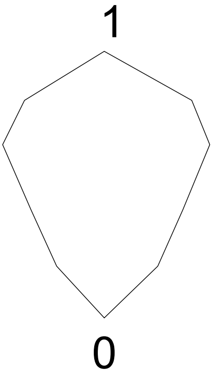
example: a simple dual neighbourly polytope



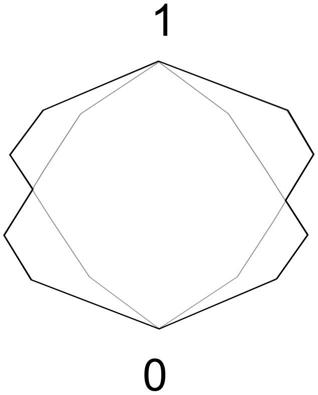
The shape of the face lattices

The face lattices of a simple and a simplicial polytope:





The connected sum



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