Minimal Clones
and
Maximal Centralizing Monoids

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Outline

Wonderful 80 !

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$$84 \quad = \quad 80 + 80 \times 0.05$$

P A R T I

In 1983, Béla Csákány determined all minimal clones on the three element set $\{0, 1, 2\}$.

B. Csákány, All minimal clones on the three element set, *Acta Cybernet.*, **6**, 1983, 227-238.

Definition

A function $f\ (\in \mathcal{O}_k)$ is called a minimal function

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- f generates a minimal clone C.
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In order to describe minimal functions, B. Csákány used the following numbering.

Csákány Numbering

A unary function $u_r(x)$ is numbered in the following way:

$$r \quad = \quad u(0) \times 3^2 + u(1) \times 3^1 + u(2) \times 3^0$$

A binary idempotent function $b_s(x, y)$ is numbered as follows:

$$b_s(x,y) \;=\; \begin{array}{|c|ccc|}\hline x\backslash y & 0 & 1 & 2 \\ \hline 0 & 0 & a & b \\ 1 & c & 1 & d \\ 2 & e & f & 2 \\ \hline \end{array}$$

$$s \;=\; a \times 3^5 + b \times 3^4 + c \times 3^3 + d \times 3^2 + e \times 3^1 + f \times 3^0$$

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$$s = a \times 3^5 + b \times 3^4 + c \times 3^3 + d \times 3^2 + e \times 3^1 + f \times 3^0$$

In other word,

$$\begin{aligned} s = \; & b(0,1) \times 3^5 + b(0,2) \times 3^4 \\ & + b(1,0) \times 3^3 + b(1,2) \times 3^2 \\ & + b(2,0) \times 3^1 + b(2,1) \times 3^0 \end{aligned}$$

Similarly, ternary majority function $m_t(x, y, z)$ is numbered as follows:

$$m_t(x,y,z) \;=\; \begin{array}{|c|c|} \hline x\backslash y & 0\;1\;2 \\ \hline 0 & 0\;0\;0 \\ 1 & 0\;1\;d \\ 2 & 0\;f\;2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline & 0\;1\;2 \\ \hline 0 & 0\;1\;b \\ 1 & 1\;1\;1 \\ 2 & e\;1\;2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline & 0\;1\;2 \\ \hline 0 & 0\;a\;2 \\ 1 & c\;1\;2 \\ 2 & 2\;2\;2 \\ \hline \end{array}$$

$$z = 0 \qquad\qquad z = 1 \qquad\qquad z = 2$$

$$t \;=\; a \times 3^5 + b \times 3^4 + c \times 3^3 + d \times 3^2 + e \times 3^1 + f \times 3^0$$

Similarly, ternary majority function $m_t(x,y,z)$ is numbered as follows:

$$m_t(x,y,z) = \underset{z=0}{\begin{array}{|c|ccc|}\hline x\backslash y & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & d \\ 2 & 0 & f & 2 \\ \hline\end{array}} \quad \underset{z=1}{\begin{array}{|c|ccc|}\hline & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & b \\ 1 & 1 & 1 & 1 \\ 2 & e & 1 & 2 \\ \hline\end{array}} \quad \underset{z=2}{\begin{array}{|c|ccc|}\hline & 0 & 1 & 2 \\ \hline 0 & 0 & a & 2 \\ 1 & c & 1 & 2 \\ 2 & 2 & 2 & 2 \\ \hline\end{array}}$$

$$t = a \times 3^5 + b \times 3^4 + c \times 3^3 + d \times 3^2 + e \times 3^1 + f \times 3^0$$

In other word,

$$\begin{aligned} t = \; & m(0,1,2) \times 3^5 + m(0,2,1) \times 3^4 \\ & + m(1,0,2) \times 3^3 + m(1,2,0) \times 3^2 \\ & + m(2,0,1) \times 3^1 + m(2,1,0) \times 3^0 \end{aligned}$$

A ternary function $p(x_1, x_2, x_3)$ is called a semiprojection if there exists $j\,(\in \{1, 2, 3\})$ such that $p(x_1, x_2, x_3) = x_j$ whenever $|\{x_1, x_2, x_3\}| < 3$.

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Semiprojection $p_t\,(x, y, z)$ is numbered in the same way:

$$t \;=\; a \times 3^5 + b \times 3^4 + c \times 3^3 + d \times 3^2 + e \times 3^1 + f \times 3^0$$

In other word,

$$\begin{aligned} t \;=\;\; & p(0,1,2) \times 3^5 + p(0,2,1) \times 3^4 \\ & + p(1,0,2) \times 3^3 + p(1,2,0) \times 3^2 \\ & + p(2,0,1) \times 3^1 + p(2,1,0) \times 3^0 \end{aligned}$$

Generators of all minimal clones on $\{0, 1, 2\}$

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(I) Unary functions (13)

$u_0 \quad u_{13} \quad u_{26}$

where

$$
\begin{aligned}
0 &= 0 \times 9 + 0 \times 3 + 0 \times 1 \\
13 &= 1 \times 9 + 1 \times 3 + 1 \times 1 \\
26 &= 2 \times 9 + 2 \times 3 + 2 \times 1
\end{aligned}
$$

and

$u_2 \quad u_{14} \quad u_8 \quad u_3 \quad u_4 \quad u_{23}$

$u_7 \quad u_{21} \quad u_{11}$

u_{15}

(II) Binary idempotent functions (48)

$$b_0 = \boxed{\begin{matrix} 0&0&0\\ 0&1&0\\ 0&0&2 \end{matrix}} \qquad b_{364} = \boxed{\begin{matrix} 0&1&1\\ 1&1&1\\ 1&1&2 \end{matrix}} \qquad b_{728} = \boxed{\begin{matrix} 0&2&2\\ 2&1&2\\ 2&2&2 \end{matrix}}$$

$$b_8 = \boxed{\begin{matrix} 0&0&0\\ 0&1&0\\ 2&2&2 \end{matrix}} \qquad b_{368} = \boxed{\begin{matrix} 0&1&1\\ 1&1&1\\ 2&2&2 \end{matrix}} \qquad b_{80} = \boxed{\begin{matrix} 0&0&0\\ 2&1&2\\ 2&2&2 \end{matrix}}$$

$$b_{36} = \boxed{\begin{matrix} 0&0&0\\ 1&1&1\\ 0&0&2 \end{matrix}} \qquad b_{40} = \boxed{\begin{matrix} 0&0&0\\ 1&1&1\\ 1&1&2 \end{matrix}} \qquad b_{692} = \boxed{\begin{matrix} 0&2&2\\ 1&1&1\\ 2&2&2 \end{matrix}}$$

---

etc. etc.

(III) Ternary majority functions (7)

$$m_0 = \boxed{\begin{matrix}0&0&0\\0&1&0\\0&0&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&0\\1&1&1\\0&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&0&2\\0&1&2\\2&2&2\end{matrix}}$$

$$m_{364} = \boxed{\begin{matrix}0&0&0\\0&1&1\\0&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&1\\1&1&1\\1&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&2\\1&1&2\\2&2&2\end{matrix}}$$

$$m_{728} = \boxed{\begin{matrix}0&0&0\\0&1&2\\0&2&2\end{matrix}} \quad \boxed{\begin{matrix}0&1&2\\1&1&1\\2&1&2\end{matrix}} \quad \boxed{\begin{matrix}0&2&2\\2&1&2\\2&2&2\end{matrix}}$$

$$m_{109} = \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{array}} \quad \boxed{\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}} \quad \boxed{\begin{array}{ccc} 0 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{array}}$$

$$m_{473} = \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{array}} \quad \boxed{\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}} \quad \boxed{\begin{array}{ccc} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{array}}$$

$$m_{510} = \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{array}} \quad \boxed{\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{array}} \quad \boxed{\begin{array}{ccc} 0 & 2 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 2 \end{array}}$$

$$m_{624} = \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}} \quad \boxed{\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}} \quad \boxed{\begin{array}{ccc} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 2 \end{array}}$$

(IV) Semiprojections (16)

$$\begin{array}{llllll} p_0 & p_{364} & p_{728} & & & \\ p_8 & p_{368} & p_{80} & p_{36} & p_{40} & p_{692} \\ p_{26} & p_{449} & p_{37} & & & \end{array}$$

and

$$\begin{array}{lll} p_{76} & p_{684} & p_{332} \\ p_{624} & & \end{array}$$

where

$$\begin{aligned} 76 &= 0 \times 3^5 + 0 \times 3^4 + 2 \times 3^3 + 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0 \\ 684 &= 2 \times 3^5 + 2 \times 3^4 + 1 \times 3^3 + 1 \times 3^2 + 0 \times 3^1 + 0 \times 3^0 \\ 332 &= 1 \times 3^5 + 1 \times 3^4 + 0 \times 3^3 + 0 \times 3^2 + 2 \times 3^1 + 2 \times 3^0 \\ 624 &= 2 \times 3^5 + 1 \times 3^4 + 2 \times 3^3 + 0 \times 3^2 + 1 \times 3^1 + 0 \times 3^0 \end{aligned}$$

Number of minimal clones on $\{0, 1, 2\}$

Unary functions	:	13	(4)
Binary idempotent functions	:	48	(12)
Ternary majority functions	:	7	(3)
Ternary semiprojections	:	16	(5)
Total	:	84	(24)

P A R T II

Centralizing Monoids

Notation

$$E_k = \{0, 1, \dots, k-1\} \quad \text{for } k > 1$$

$\mathcal{O}_k^{(n)} \left(= E_k^{(E_k)^n}\right)$: The set of n-variable functions on E_k

$$\mathcal{O}_k = \bigcup_{n=1}^{\infty} \mathcal{O}_k^{(n)}$$

$\mathcal{J}_k$: The set of all projections e_i^n $(1 \leq i \leq n)$ on E_k

where $e_i^n(x_1, \dots, x_i, \dots, x_n) = x_i$

for $\forall\, x_1, \dots, x_n \in E_k$

Definition

For $f \in \mathcal{O}_k^{(m)}$ and $g \in \mathcal{O}_k^{(n)}$
f and g commute (expressed as $f \perp g$)
if the following holds for every $m \times n$ matrix $A = (x_{ij})$ over E_k

$$\begin{aligned} & f(g(x_{11},\ldots,x_{1n}),\ldots,g(x_{m1},\ldots,x_{mn})) \\ = \; & g(f(x_{11},\ldots,x_{m1}),\ldots,f(x_{1n},\ldots,x_{mn})) \end{aligned}$$

$$\begin{array}{cccc|c} x_{11} & x_{12} & \cdots & x_{1n} & g(\ldots,x_{1j},\ldots) \\ x_{21} & x_{22} & \cdots & x_{2n} & g(\ldots,x_{2j},\ldots) \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} & g(\ldots,x_{m1},\ldots) \\ \hline f(\ldots,x_{i1},\ldots) & f(\ldots,x_{i2},\ldots) & \cdots & f(\ldots,x_{in},\ldots) & f(g,\ldots) = g(f,\ldots) \end{array}$$

Example

- $f \in \mathcal{O}_k^{(m)}$: constant function
- $g \in \mathcal{O}_k^{(n)}$: idempotent function

$\Longrightarrow$

f and g commute, i.e. , $f \perp g$

(Here, g is *idempotent* if

$$g(a, \ldots, a) = a \quad \text{for} \quad \forall\, a \in E_k.)$$

The case where f is a unary function :

For $f \in \mathcal{O}_k^{(1)}$ and $g \in \mathcal{O}_k^{(n)}$

f and g commute ($f \perp g$)

if the following holds for all $(b_1, \ldots, b_n) \in (E_k)^n$

$$f\,(\,g\,(b_1, \ldots, b_n)\,) \;=\; g\,(\,f\,(b_1), \ldots, f\,(b_n)\,)$$

Centralizer

Definition

For $F \subseteq \mathcal{O}_k$ define F^* by

$$F^* = \{ g \in \mathcal{O}_k \mid g \perp f \text{ for all } f \in F \}$$

F^* is called the centralizer of F.

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F^* is called the centralizer of F.

Note: A centralizer is always a clone.

Centralizing Monoids

A "centralizing monoid" can be characterized

in several different ways.

Lemma

For $M \subseteq \mathcal{O}_k^{(1)}$, the following conditions are equivalent.

(1) $M = M^{**} \cap \mathcal{O}_k^{(1)}$

(2) $\exists\, F \subseteq \mathcal{O}_k$, $M = F^* \cap \mathcal{O}_k^{(1)}$

(M is the unary part of some centralizer)

(3) $\exists\, \mathcal{A} = (E_k;\ F)$: algebra, $M = \mathrm{End}\,(\mathcal{A})$

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(3) $\exists\, \mathcal{A} = (E_k;\ F)$: algebra, $M = \mathrm{End}\,(\mathcal{A})$

Definition

For $M \subseteq \mathcal{O}_k^{(1)}$, M is a **centralizing monoid** if M satisfies the above conditions.

Witness

Lemma (Witness Lemma)

For a monoid $M \subseteq \mathcal{O}^{(1)}$ of unary functions
and a subset $S \subseteq \mathcal{O}$,

suppose the following conditions (i) and (ii) hold:

(i) For $\forall f \in M$ and $\forall u \in S$ $\quad f \perp u$

(ii) For $\forall g \in \mathcal{O}^{(1)} \setminus M$ and $\exists w \in S$ $\quad g \not\perp w$

Then M is a centralizing monoid.

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Then M is a centralizing monoid.

Definition

We say that

S in the lemma is a witness for a centralizing monoid M.

Notation and Property of Witnesses

Notation

Denote by $M(S)$ the centralizing monoid M which has S as its witness. (i.e., $M(S) = S^* \cap \mathcal{O}_k^{(1)}$)

Lemma

Every centralizing monoid M has a witness.

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Proof M^* is a witness for M. $\square$

Theorem

For every centralizing monoid M there exists a finite subset of $\mathcal{O}_k$ which is a witness of M,

that is,

every centralizing monoid M has a finite witness.

Proof. Let $S\,(\subseteq \mathcal{O}_k)$ be a witness for M.
For each $f \in \mathcal{O}_k^{(1)} \setminus M$ there exists $u \in S$ such that $f \not\perp u$.
We pick one such $u\,(= u_f)$ for each f and let

$$T = \{\, u_f \mid f \in \mathcal{O}_k^{(1)} \setminus M \,\}.$$

Then, T is clearly a witness for M.
Furthermore, T is finite because $\mathcal{O}_k^{(1)}$ is finite. $\square$

Now we turn to maximal centralizing monoids,

which are related to minimal clones !!

Definition

A centralizing monoid M is maximal if $\mathcal{O}_k^{(1)}$ is the only centralizing monoid properly containing M.

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Theorem

For any maximal centralizing monoid M, there exists $u \, (\in \mathcal{O}_k)$ such that

$$M = M(u),$$

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(**Proof** $\quad M(S_1) \cap M(S_2) = M(S_1 \cup S_2)$)

Theorem

For any maximal centralizing monoid M, there exists a minimal function $f\,(\in \mathcal{O}_k)$ such that

$$M = M(f)\,,$$

that is,

every maximal centralizing monoid has a witness which is a minimal function.

Proof

- Since a maximal centralizing monoid has a singleton witness, there exists $g \in \mathcal{O}_k$ such that $M = M(g)$.

- Every non-trivial clone C (i.e., $C \neq \mathcal{J}_k$) contains a minimal clone. Hence, there exists $f \in \mathcal{O}_k$ which satisfies the following.

  (i) $\langle f \rangle$ is a minimal clone.
  (ii) $\langle f \rangle \subseteq \langle g \rangle \quad (\Leftrightarrow \; f \in \langle g \rangle)$

- In general, for any $u, v, w \in \mathcal{O}_k$,

$$u \in \langle v \rangle \text{ and } v \perp w \quad \Longrightarrow \quad u \perp w.$$

As a corollary,

$$u \in \langle v \rangle \quad \Longrightarrow \quad v^* \subseteq u^*.$$

Proof (cont.)

- Hence, for f and g given above,

$$g^* \subseteq f^*$$

- It follows that

$$M(g) = g^* \cap \mathcal{O}_k^{(1)} \subseteq f^* \cap \mathcal{O}_k^{(1)} = M(f)$$

- Since $M(g)$ is a maximal centralizing monoid, by assumption, it holds either

$$M(g) = M(f) \quad (\Rightarrow M = M(f))$$

or

$$M(f) = \mathcal{O}_k^{(1)}.$$

Proof (cont. cont.)

- However, we know that

$$(S_k \cup \mathrm{Const})^* \;=\; J_k\,.$$

- Therefore, $M(f) \;=\; \mathcal{O}_k^{(1)}$ cannot happen for a minimal function f, and so

$$M \;=\; M(f)$$

  must hold.

This completes the proof. $\square$

P A R T III

Minimal Clones and Maximal Centralizing Monoids on $E_3 = \{0, 1, 2\}$

From here, we shall concentrate on the ternary case,

that is, the case where the base set is

$$E_3 = \{ 0, 1, 2 \}.$$

Centralizing Monoids on $\{0, 1, 2\}$

First,

we have determined all maximal centralizing monoids on E_3 using the result on minimal clones due to B. Csákány.

The number of the maximal centralizing monoids is 10.

Then

we have enumerated all centralizing monoids on E_3.

The number of the centralizing monoids is 192.

Question (concerning all centralizing monoids)

- The number of centralizing monoids on E_3 is 192.
- They are divided into 48 conjugate classes.

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Both numbers are beautiful numbers !!

$$192 = 2^6 \times 3 \quad \text{and} \quad 48 = 2^4 \times 3$$

Question (concerning all centralizing monoids)

- The number of centralizing monoids on E_3 is 192.
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Is this phenomenon just for $k = 3$?

Or, could this be generalized to any $\boldsymbol{k}$ (> 3) ??

By the way, what is the number 48 ?

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2. It is the number of minimal clones generated by binary idempotent functions on E_3.

By the way, what is the number 48 ?

1. It is the number of conjugate classes of centralizing monoids on E_3.
2. It is the number of minimal clones generated by binary idempotent functions on E_3.
3. It is also the age of the Chairperson of this Session !?
(Addition of 20 % Tax needed ???)

Maximal Centralizing Monoids on $\{0, 1, 2\}$

For each minimal function $f \in \mathcal{O}_3^{(1)}$, let $\{ f \}$ be a witness and determine a centralizing monoid $M(f)$.

Then,

some of such centralizing monoids are maximal, while some are not maximal.

Maximal Centralizing Monoids on $\{0, 1, 2\}$

For each minimal function $f \in \mathcal{O}_3^{(1)}$, let $\{ f \}$ be a witness and determine a centralizing monoid $M(f)$.

Then,

some of such centralizing monoids are maximal, while some are not maximal.

IMPORTANT: All maximal centralizing monoids can be obtained in this way.

Proposition

Over a three-element set, there are 10 maximal centralizing monoids.

More precisely:

- there are 3 maximal centralizing monoids, each of which has a unary constant function as its witness.
- there are 7 maximal centralizing monoids, each of which has a ternary majority function which generates a minimal clone as its witness.

The following is the list of minimal functions which, as witnesses, correspond to maximal centralizing monoids:

Constant functions

$$\begin{aligned} c_0(x) &= 0 \\ c_1(x) &= 1 \\ c_2(x) &= 2 \end{aligned}$$

Majority functions

(showing the values only for $|\{x,y,z\}| = 3$)

$$m_0(x,y,z) = 0 \quad \text{if} \quad |\{x,y,z\}| = 3$$
$$m_{364}(x,y,z) = 1 \quad \text{if} \quad |\{x,y,z\}| = 3$$
$$m_{728}(x,y,z) = 2 \quad \text{if} \quad |\{x,y,z\}| = 3$$

$$m_{109}(x,y,z) = \begin{cases} 0 & \text{if} \quad (x,y,z) \in \sigma \\ 1 & \text{if} \quad (x,y,z) \in \tau \end{cases}$$

$$m_{473}(x,y,z) = \begin{cases} 1 & \text{if} \quad (x,y,z) \in \sigma \\ 2 & \text{if} \quad (x,y,z) \in \tau \end{cases}$$

$$m_{510}(x,y,z) = \begin{cases} 2 & \text{if} \quad (x,y,z) \in \sigma \\ 0 & \text{if} \quad (x,y,z) \in \tau \end{cases}$$

$$m_{624}(x,y,z) = y \quad \text{if} \quad |\{x,y,z\}| = 3$$

where $\sigma = \{0,1,2),(1,2,0),(2,0,1)\}$
and $\tau = \{(0,2,1),(1,0,2),(2,1,0)\}$

Number of Elements in Maximal Centralizing Monoids on $\{0,1,2\}$

$M(c_0)$	9
$M(c_1)$	9
$M(c_2)$	9
$M(m_0)$	17
$M(m_{364})$	17
$M(m_{728})$	17
$M(m_{109})$	11
$M(m_{473})$	11
$M(m_{510})$	11
$M(m_{624})$	9

Remark :

There exist other minimal functions which serve as witnesses of maximal centralizing monoids.

They are:

Binary function: b_{624}

and

Semiprojections: p_{76}, p_{684}, p_{332} and p_{624}

However, the centralizing monoids having them as witnesses all coincide with already known centralizing monoids.

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More precisely,

Binary function: $M(b_{624}) = M(m_{624})$

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More precisely,

Binary function: $M(b_{624}) = M(m_{624})$

and

Semiprojections:
$$\begin{aligned} M(p_{76}) &= M(m_{473}) \\ M(p_{684}) &= M(m_{510}) \\ M(p_{332}) &= M(m_{109}) \\ M(p_{624}) &= M(m_{624}) \end{aligned}$$

Just for curiosity,

$$m_{624} = \underset{z=0}{\boxed{\begin{matrix}0&0&0\\0&1&2\\0&1&2\end{matrix}}} \quad \underset{z=1}{\boxed{\begin{matrix}0&1&2\\1&1&1\\0&1&2\end{matrix}}} \quad \underset{z=2}{\boxed{\begin{matrix}0&1&2\\0&1&2\\2&2&2\end{matrix}}}$$

$$b_{624} = \boxed{\begin{matrix}0&2&1\\2&1&0\\1&0&2\end{matrix}}$$

$$p_{624} = \underset{z=0}{\boxed{\begin{matrix}0&0&0\\1&1&2\\2&1&2\end{matrix}}} \quad \underset{z=1}{\boxed{\begin{matrix}0&0&2\\1&1&1\\0&2&2\end{matrix}}} \quad \underset{z=2}{\boxed{\begin{matrix}0&1&0\\0&1&1\\2&2&2\end{matrix}}}$$

Remark :

One can generalize the results on constant functions for any $k > 1$.

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Theorem

For any $k > 1$ and any constant function c on E_k,

$M(c)$ is a maximal centralizing monoid.

Proof

We assume $c = c_0$, the constant function taking value 0.

Lemma 1

$$M(c_0) \;=\; (\mathrm{Pol}(0))^{(1)}$$

Lemma 2

$$(\mathrm{CONST})^* \;=\; \mathrm{IDEMP}$$

Lemma 3

For $f \in \mathcal{O}_k$, if $f \in (\mathrm{Pol}(0)^{(1)})^* \cap \mathrm{IDEMP}$ then f is conservative.

Lemma 4

$$(\mathrm{Pol}(0)^{(1)})^* \;\cap\; \mathrm{IDEMP} \;=\; \mathcal{J}_k$$

(This lemma follows from several Claims.)

Proof of Theorem For any $u \in \mathcal{O}_k^{(1)} \setminus M(c_0)$ let M be a monoid containing $M(c_0) \cup \{u\}$. Since $M(c_0) = \mathrm{Pol}(0)^{(1)}$, u maps 0 to some $a \neq 0$. Then M must necessarily contain all constant functions. Hence we have

$$M \supset M(c_0) \cup \mathrm{CONST}.$$

It follows that

$$M^* \subseteq M(c_0)^* \cap \mathrm{CONST}^* .$$

which implies, by Lemmas 1 and 2, that

$$M^* \subseteq (\mathrm{Pol}(0)^{(1)})^* \cap \mathrm{IDEMP}.$$

Since M^* is a clone and contains $\mathcal{J}_k$ it follows by Lemma 4 that $M^* = \mathcal{J}_k$. By applying $*$ to both sides, we obtain

$$M^{**} \; = \; \mathcal{J}_k^* \, (= \; \mathcal{O}_k).$$

Hence

$$M^{**} \cap \mathcal{O}_k^{(1)} \; = \; \mathcal{O}_k^{(1)}.$$

Therefore, if M is a centralizing monoid then, by definition,
$M \, (= M^{**} \cap \mathcal{O}_k^{(1)}) \; = \; \mathcal{O}_k^{(1)}$.
This concludes that $M(c_0)$ is a maximal centralizing monoid. $\square$

Problem

Is it also possible to generalize the results on minimal majority functions?

Namely, is it true that,

for every majority function f, if f is minimal

then $M(f)$ is a maximal centralizing monoid ?

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Still open.

Hopefully,
the answer will be reported at the Conference celebrating the "84" th Birthday of Béla Csákány !!

Thank you

for your attention !

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and

Thank you, Béla

for your great contribution and friendship
to Algebra community in the world !!