Optimal strong Mal'cev conditions implying congruence meet semi–distributivity

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Jelena Jovanović
University of Belgrade

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Tame congruence theory types

In this talk we aim to find strong Mal'cev conditions for characterizing congruence meet semi–distributivity in locally finite varieties by two at most ternary terms (these would be the optimal strong Mal'cev conditions). However, we only manage to find three systems of identities that are the only candidates for this.

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- *boolean type*
- *lattice type*
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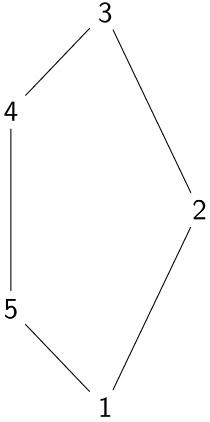
For locally finite varieties omitting certain types can be characterized by Mal'cev conditions:

The lattice of tame congruence theory types



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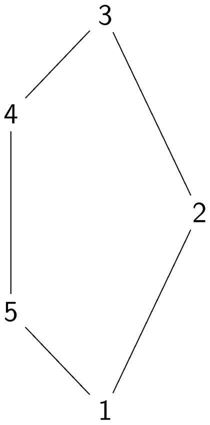

Omitting each order ideal of this lattice of types is characterized by a Mal'cev condition.
In particular, omitting ideals $\{1\}$ and $\{1,2\}$ is characterized by strong Mal'cev conditions.

Strong Mal'cev conditions

Definitions

By a *strong Mal'cev condition* we mean a finite set of identities such that each of its function symbols is interpreted as a term operation instead of a fundamental operation.
A strong Mal'cev condition Σ is *idempotent* if all of its function symbols are (that is, $\Sigma \models f(x, x, \ldots, x) \approx x$ for each function symbol f appearing in Σ).
A term t is linear if there is at most one function symbol in it.
An identity $s \approx t$ is linear if both s and t are linear.

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All the strong Mal'cev conditions considered here are both idempotent and linear.

Optimal strong Mal'cev conditions for describing omitting type 1

Theorem (Kearnes, Marković, McKenzie)

A locally finite variety $\mathcal{V}$ omits type 1 iff there exists a 4–ary term t such that:
(1) $\mathcal{V} \models t(x,x,x,x) \approx x$, *and*
(2) $\mathcal{V} \models t(x,y,z,y) \approx t(y,z,x,x)$.

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Corollary (Kearnes, Marković, McKenzie)

A locally finite variety $\mathcal{V}$ omits type 1 iff there exists a 4–ary term e such that:
(1) $e(x,x,x,x) \approx x$, and
(2) $e(y,y,x,x) \approx e(y,x,y,x) \approx e(x,x,x,y)$ (this term is called a weak 3–edge term).

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Theorem (Kozik)

A locally finite variety $\mathcal{V}$ satisfies congruence $SD(\wedge)$ iff is has 3–ary and 4–ary weak near–unanimity terms, v and w respectively, that satisfy the identity $v(y, x, x) \approx w(y, x, x, x)$.

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Corollary (Maroti, Janko)

A locally finite variety $\mathcal{V}$ satisfies congruence $SD(\wedge)$ iff is has a 3–ary weak near–unanimity term s and 3–ary terms r and t satisfying

$$r(x,x,y) \approx r(x,y,x) \approx t(y,x,x) \approx t(x,y,x) \approx s(x,x,y)$$
$$r(y,x,x) \approx t(y,y,x)$$

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Example (1)

Let $\mathbf{A}$ be a finite algebra with at least two elements and a single idempotent basic operation $f(x_1, x_2, x_3)$, which is a majority term:

$$f(x,x,y) \approx f(x,y,x) \approx f(y,x,x) \approx x.$$

In case no arguments are equal we can define f like this:
$f(a,b,c) = a$, for all a, b, c in $\mathbf{A}$ and $a \neq b$, $b \neq c$, $c \neq a$.

This algebra omits types 1 and 2 (Kozik's theorem stated earlier).

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This algebra omits types 1 and 2 (Kozik's theorem stated earlier). It also has some interesting properties:

- the only binary terms in **A** are projections π_1 and π_2
- every ternary term in **A** is either some of the π_1,π_2, π_3 or a majority term

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Theorem

Let $\mathbf{A}$ be a finite idempotent algebra and $\mathcal{V}$ the variety generated by $\mathbf{A}$. Then $\mathcal{V}$ satisfies congruence $SD(\wedge)$ iff it does not contain an algebra that is term equivalent to a full idempotent reduct of a module over some finite ring.

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Let $\mathbf{A}$ *be a finite idempotent algebra and* $\mathcal{V}$ *the variety generated by* $\mathbf{A}$*. Then* $\mathcal{V}$ *satisfies congruence* $SD(\wedge)$ *iff it does not contain an algebra that is term equivalent to a full idempotent reduct of a module over some finite ring.*

So, the system we are looking for has to hold in algebras $\mathbf{A}$ and $\mathbf{B}$, and it must not hold in any full idempotent reduct of a module over a finite ring.

Lemma (Kearnes, Marković, McKenzie)

Let Σ be an idempotent, linear, strong Malcev condition in a language L, and let Σ_0 be the set of all linear consequences of Σ that involve no function symbols of arity strictly less than 3. Either
(1) Σ and Σ_0 are realized by the same varieties, or
(2) $\Sigma \models q(x, y) \approx q(y, x)$ for some binary function symbol q of L.
(Or both.)

Systems of identities involving terms of arities less than 3

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Corollary

Since algebra $\mathbf{A}$ does not have a commutative binary term, it is sufficient to examine only systems on one or two ternary terms.

Fact

A system of identities on a single ternary term cannot characterize congruence SD($\wedge$)*, for if it holds in algebras* $\mathbf{A}$ *and* $\mathbf{B}$*, it also holds in a full idempotent reduct of a module over* $\mathbb{Z}_5$*.*

Systems of identities on two ternary terms

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- both p and q are majority terms in $\mathbf{A}$: $x \approx p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx q(x,x,y) \approx q(x,y,x) \approx q(y,x,x)$

Systems of identities on two ternary terms possibly characterizing $SD(\wedge)$

From the previous two systems, by eliminating identities, we obtain systems that are candidates for characterizing congruence $SD(\wedge)$. The case analysis is rather long and tedious, but we end up with these three systems:

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Systems that possibly describe congruence $SD(\wedge)$

$$\begin{cases} p(x,x,y) \approx p(x,y,y) \\ p(x,y,x) \approx q(x,x,y) \approx q(x,y,x) \approx q(y,x,x) \end{cases} \tag{1}$$

$$\begin{cases} x \approx q(x,y,x) \\ p(x,y,y) \approx p(x,y,x) \\ p(x,x,y) \approx q(x,x,y) \approx q(y,x,x) \end{cases} \tag{2}$$

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Conclusions

Proposition

Each of the systems (1), (2),(3) *implies and possibly characterizes congruence meet–semidistributivity. Furtermore, if it is possible to describe this property by two ternary terms, it can only be done by one or more of these systems.*

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AN OPEN PROBLEM: does any of the systems (1), (2), (3) actually describe congruence $SD(\wedge)$?