# On axiomatisablity questions about monoid acts


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# Monoid acts

## Right acts

$A$ is a left $S$-act if there exists a map $\cdot\colon S \times A \longrightarrow A$ such that for every $s, t \in S$ and $a \in A$,

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## $S$-morphisms

Let $A, B$ be left $S$-acts. The map $\varphi\colon A \to B$ is an $S$-morphism if

$$(s \cdot a)\varphi = s \cdot a\varphi$$

for all $a \in A$ and $s \in S$.

# Some classes of acts

## Free acts

A left $S$-act $F$ is *free* over a set $X$ if there exists a map $\iota\colon X \to F$ such that for every map $\varphi\colon X \to A$ into a left $S$-act $A$, there exists a unique $S$-morphism $\psi\colon F \to A$ satisfying $\iota\psi = \varphi$.

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### Projective acts

A left $S$-act $P$ is *projective* if, for every $S$-morphism $\varphi\colon P \to B$ and surjective $S$-morphism $\psi\colon A \to B$, there exists an $S$-morphism $\chi\colon P \to A$ such that $\varphi = \chi\psi$.

# Strongly flat acts

**Definition**

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## B. Stenström, 1970

The following are equivalent for a left $S$-act $A$:

- $A$ is strongly flat,
- $A$ is a direct limit of finitely generated free left $S$-acts,
- $A$ satisfies the following conditions.
  - (P) For all $a, a' \in A$ and $s, s' \in S$ if $sa = s'a'$ then there exist $a'' \in A$ and $u, u' \in S$ such that $a = ua''$, $a' = u'a''$ and $su = s'u'$.
  - (E) For all $a \in A$ and $s, s' \in S$ if $sa = s'a$ then there exists $a'' \in A$ and $u \in S$ such that $a = ua''$ and $su = s'u$.

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# Axiomatisability

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Given a monoid $S$, is the class of all strongly flat left $S$-acts first-order axiomatisable?
The first-order language contains the one-variable function symbols $\{\lambda_s : s \in S\}$, and the equality relation.

**Answer, Gould, 1987**

The class of strongly flat left $S$-acts is axiomatisable if and only if the following two conditions hold:

**(R)** For every $s, t \in S$, the subact

$$\mathbf{R}(s,t) = \{(u,v) : su = tv\} \subseteq S \times S$$

is finitely generated, that is, there exists a finite set $X \subseteq \mathbf{R}(s,t)$ such that $\mathbf{R}(s,t) = X \cdot S$.

**(r)** For every $s, t \in S$, the right ideal

$$\mathbf{r}(s,t) = \{u : su = tu\} \subseteq S$$

is finitely generated.

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# Axiomatisability and constructions

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## Submonoid, homomorphic image

The class of monoids satisfying (**R**) is not closed under taking homomorphic images or submonoids.
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**(R) implies (r)**

If $Y$ is a semilattice, and it satisfies (**R**), then it satisfies (**r**).
The converse is not true.

# Semilattices

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## Proof:

If $\mathbf{R}(\alpha, \beta) = X \cdot Y$, then

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If $Y$ is a semilattice, and it satisfies (**R**), then it is finite above: every principal filter is finite.

If $Y$ is a semilattice that satisfies (**R**), then it is a lattice.

# Lattices

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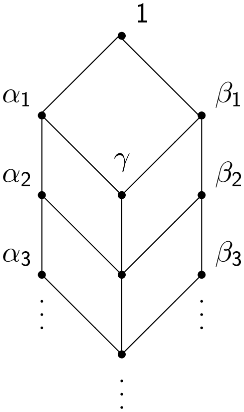
### (**R**) and noetherian properties

There exists a semilattice which is finite above, it has width 3, but it does not satisfy (**R**). All of its ideals are finitely generated.

# An example



## A reformulation

A distributive lattice $Y$ satisfies (**r**) if and only if it has a 'generalised dual symmetric difference', that is, for every $\alpha, \beta \in Y$, the set

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# Distributive lattices and (**r**)

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has a greatest element.

Both Boolean lattices and completely distributive lattices satisfy (**r**).

# Example

Let $Y$ be the sublattice (not complete sublattice) of $\mathcal{P}(\mathbb{N})$ generated by the following sets, together with $\mathbb{N}$:

$$A = \{n : n \equiv 1 \pmod 3\}$$
$$B = \{n : n \equiv 2 \pmod 3\}$$
$$C_i = \{n : 3 \mid n, n \leq 3i\} \text{ for all } i \geq 1$$

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Then the lattice $Y$ is a distributive lattice such that $(Y, \cap)$ does not satisfy (**r**).

Because $\mathbf{r}(A, B) = \{C_i : i = 1, 2, \ldots\}$, which is not finitely generated.

# Clifford monoids

## Definition

Let $Y$ be a semilattice, and for every $\alpha \in Y$, let $G_\alpha$ be a group with identity $e_\alpha$. Furthermore, for every $\alpha \geq \beta$, let $\varphi_{\alpha,\beta}\colon G_\alpha \to G_\beta$ be a homomorphism such that $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ for every $\alpha \geq \beta \geq \gamma$. Let $S = \cup_{\alpha \in Y} G_\alpha$, and define a multiplication on $S$ by

$$s \cdot t = s\varphi_{\alpha,\alpha\beta} \cdot t\varphi_{\beta,\alpha\beta},$$

where $s \in G_\alpha$ and $t \in G_\beta$.
$S$ is an inverse semigroup with semilattice of idempotents $E(S) = \{e_\alpha : \alpha \in Y\}$ isomorphic to $Y$. $E(S)$ is a retract of $S$. $S$ is a monoid if and only if $Y$ is a monoid.

**(r)**

If $Y$ is finite, then $S$ satisfies (**r**).

# Clifford monoids with finite semilattice of idempotents

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**Proof:**

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### (R)

If $Y$ is finite, then $S$ satisfies (**R**) if and only if for every $\alpha, \beta, \gamma \in Y$ satisfying $\gamma \leq \alpha \wedge \beta$, we have that $[H : K]$ is finite where

$$H = \{(u, v) : u\varphi_{\alpha,\gamma} = v\varphi_{\beta,\gamma}\} \leq G_\alpha \times G_\beta$$
$$K = \{(g\varphi_{\alpha\vee\beta,\alpha}, g\varphi_{\alpha\vee\beta,\beta}\} \leq G_\alpha \times G_\beta.$$

# Clifford monoids with infinite semilattice of idempotents and trivial structure homomorphisms

**Definition**

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**Theorem**

$S$ satisfies $(\mathbf{R})$ if and only if the following are true

1. $Y$ is finite above,
2. for every $0 \neq \alpha \in Y$, $G_\alpha$ is finite,
3. for every $\alpha \in Y$, the set $\{\beta : \beta \perp \alpha, |G_\beta| > 1\}$ is finite,
4. for every $s \in G_\alpha$ and $t \in G_\beta$, $\mathbf{R}(s,t)\mathcal{J}$ is finitely generated.