Lattices being blocks of skeleton tolerances

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Content

Tolerances

Let L be a lattice. A binary relation T on L is called a tolerance relation iff it is reflexive, symmetric and compatible with joins and meets of the lattice. We define a block of T as a maximal subset of L in which every two elements are in the relation T.

Tolerances

Let L be a lattice. A binary relation T on L is called a tolerance relation iff it is reflexive, symmetric and compatible with joins and meets of the lattice. We define a block of T as a maximal subset of L in which every two elements are in the relation T.

Every block of a tolerance relation T on a finite lattice L is an interval of L and the set L/T of all blocks of R in L with an order defined by

$$\alpha \leq \beta \Longleftrightarrow 0_\alpha \leq 0_\beta$$

(which is equivalent to the fact that $1_\alpha \leq 1_\beta$), where $\alpha = [0_\alpha, 1_\alpha]$, $\beta = [0_\beta, 1_\beta]$ are blocks of T, forms a lattice called the factor lattice of L by T.

Skeleton

A tolerance whose transitive closure is the total relation on L is called a glued tolerance.
The intersection of any number of glued tolerances is a glued tolerance.

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The skeleton tolerance of a lattice L, which will be denoted by $\Sigma(L)$ (or just Σ when the context is obvious), is the smallest glued tolerance of L. The factor lattice $L/\Sigma(L)$ is called the skeleton of L and denoted by $S(L)$.

Blocks and skeletons in finite distributive lattices

If L is a finite modular lattice, then blocks of it skeleton tolerance relation $\Sigma(L)$ are maximal complementary intervals of the lattice. In particular, if L is a finite distributive lattice, then the blocks are maximal Boolean intervals of L.

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Theorem

Every finite lattice is a skeleton of some finite distributive lattice.

Skeleton tolerances

The skeleton tolerance of a lattice L is generated by the set of all prime quotients in L, i.e., pairs (a, b) such that $a \prec b$.

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Lemma

If $\alpha = [0_\alpha, 1_\alpha]$ is a block of $\Sigma(L)$ for some finite lattice L, then $0_\alpha = p(x_1, \ldots, x_n)$ and $1_\alpha = p(y_1, \ldots, y_n)$ for a lattice polynomial p and a system $x_1, \ldots, x_n, y_1, \ldots, y_n$ of elements of L such that $x_j \preceq y_j$ for every $j = 1, \ldots, n$.

Two-tailed lattices are not blocks of a skeleton tolerance. . .

Let $M(L)$ and $J(L)$ denote, respectively, sets of all meet-irreducible and join-irreducible elements of the lattice L.

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Lemma

Let L be a finite lattice. If $\alpha = [0_\alpha, 1_\alpha]$ is a block of $\Sigma(L)$ such that $0_\alpha \nprec 1_\alpha$. Then $0_\alpha \notin M(L)$ or $1_\alpha \notin J(L)$.

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Let L be a finite lattice. If $\alpha = [0_\alpha, 1_\alpha]$ is a block of $\Sigma(L)$ such that $0_\alpha \not\prec 1_\alpha$. Then $0_\alpha \notin M(L)$ or $1_\alpha \notin J(L)$.

Corollary

No chain of length greater than 1 can be a block of $\Sigma(L)$ for any lattice L.

. . . but all others are

Theorem

Let L be a finite lattice. L is a block of the skeleton tolerance of a finite lattice iff $|L| \leq 2$ or $0_L \notin M(L)$ or $1_L \notin J(L)$.

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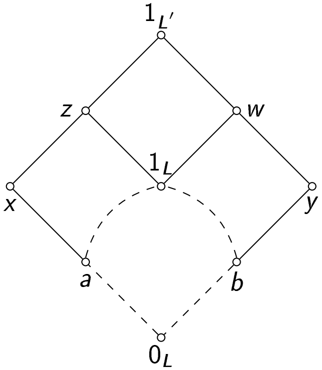


Blocks belonging to the same block overlap

Lemma

Let L be a finite lattice. Assume that $\alpha, \beta \in S(L)$ such that $(\alpha, \beta) \in \Sigma(S(L))$. Then $\alpha \cap \beta \neq \emptyset$.

Corollary

If $[\alpha, \beta] \in S(S(L))$, then $\alpha \cap \beta \neq \emptyset$.

Important embeddings

Theorem

If L is a finite lattice, then every block $[\alpha, \beta]$ of $\Sigma(S(L))$ can be join-embedded in α and meet-embedded in β.

When the embeddings are possible?

Lemma

Let L be a finite lattice, D be a distributive lattice. If there is a meet-embedding $\pi\colon L \to D$ then $|J(D)| \geq |J(L)|$. Dually, if there is a join-embedding $\sigma\colon L \to D$ then $|M(D)| \geq |M(L)|$.

When the embeddings are possible?

Lemma

Let L be a finite lattice, D be a distributive lattice. If there is a meet-embedding $\pi\colon L \to D$ then $|J(D)| \geq |J(L)|$. Dually, if there is a join-embedding $\sigma\colon L \to D$ then $|M(D)| \geq |M(L)|$.

If D is a finite Boolean lattice then

$$\dim D = |J(D)| = |M(D)|.$$

Thus, if π is a meet-embedding of a finite lattice L into a Boolean lattice α then $\dim \alpha \geq |J(L)|$ and, similarly, if σ is a join-embedding of L into a Boolean lattice β then $\dim \beta \geq |M(L)|$.

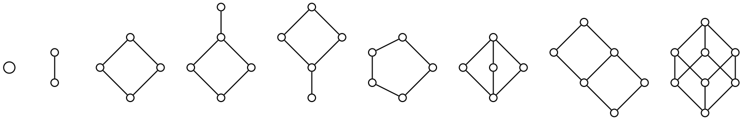
Blocks of skeletons of distributive lattices with at most k-dimensional Boolean cubes

A block of the skeleton of a distributive lattice with at most k-dimensional maximal Boolean intervals

- can be join- and meet-embedded into a k-dimensional Boolean lattice, i.e., the number of its join- and meet-irreducible elements is not greater than k;
- cannot be a two-tailed lattice.

The case $k=3$

Possible blocks and the number of possible non-isomorphic distributive lattices for these blocks.



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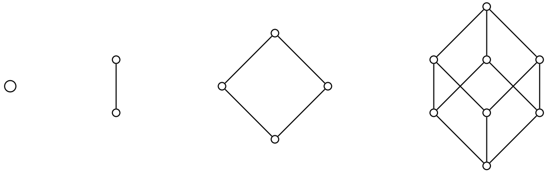
Possible non-isomorphic distributive lattices for ○



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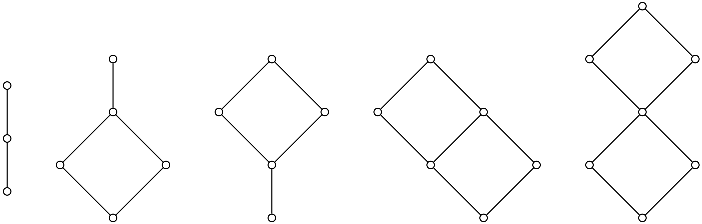

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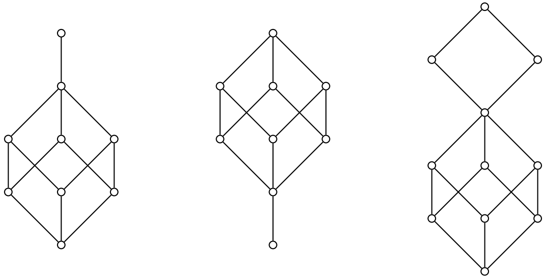

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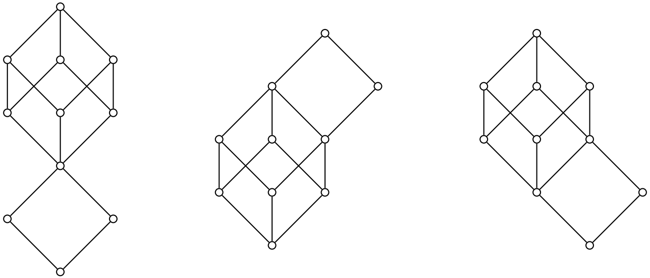

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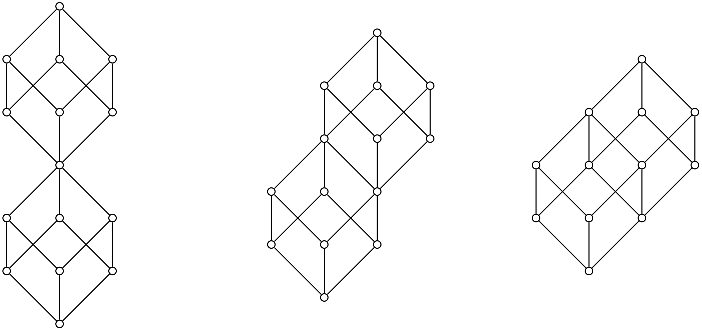
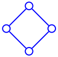
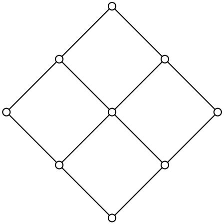
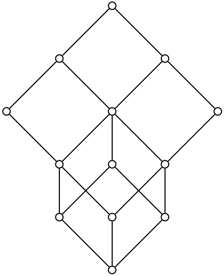
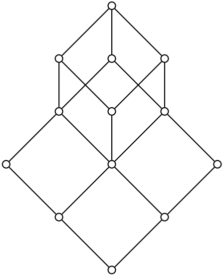
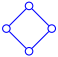


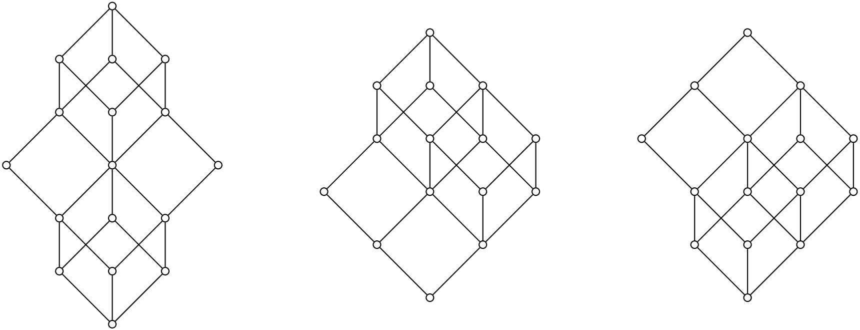
Possible non-isomorphic distributive lattices for 

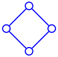


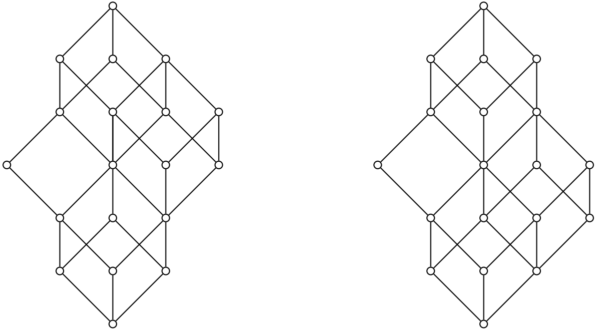


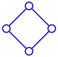


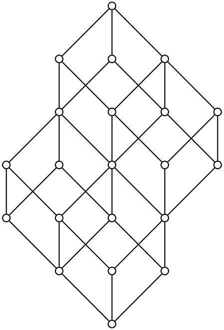
Possible non-isomorphic distributive lattices for 

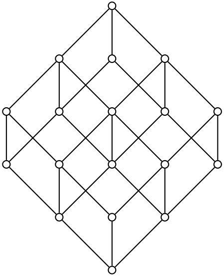


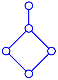
Possible non-isomorphic distributive lattices for 

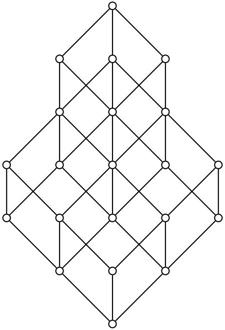


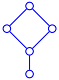
Possible non-isomorphic distributive lattices for 

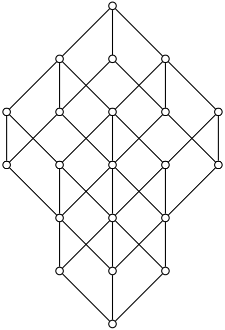


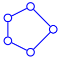


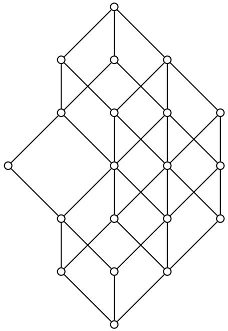
Possible non-isomorphic distributive lattices for 

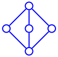


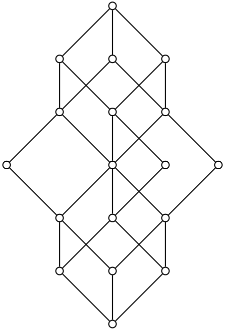
Possible non-isomorphic distributive lattices for 

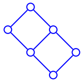


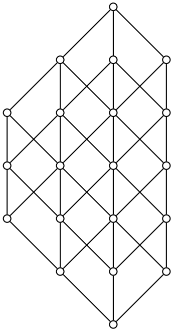
Possible non-isomorphic distributive lattices for 

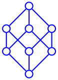


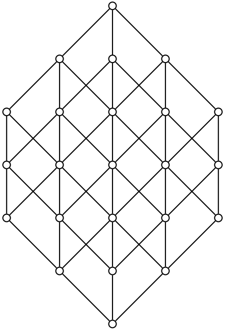
Possible non-isomorphic distributive lattices for 



Possible non-isomorphic distributive lattices for 



Possible non-isomorphic distributive lattices for 



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Thank you for your attention.