On the tolerance lattice of tolerance factors, II

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In this talk we are going to formulate analogous results for tolerance factors of lattices.

The "fitting into" order

If φ, θ are equivalence relations and $\varphi \leq \theta$, then any equivalence class of θ is a union of some equivalence classes of φ. The same is true for the congruences of an algebra $\mathbb{A} = (A, F)$ (see the figure below).

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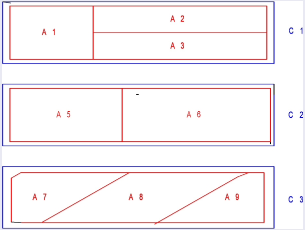

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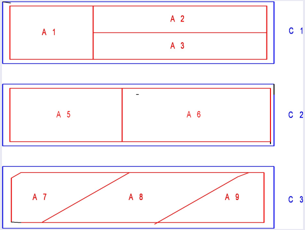

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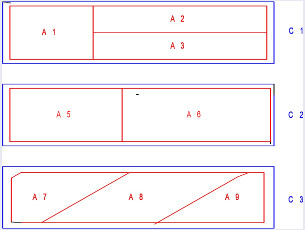

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We already know that $\sqsubseteq$ *is a partial order* on $\mathrm{Tol}(L)$.

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This yields also $x \leq \mu(x)$, for all $x \in L$. If (σ, μ) is a polarity, then σ is a join-homomorphism and μ is a meet-homomorphism of $\mathbb{L}$, i.e. for all $x_1, x_2 \in L$

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In the case of a finite lattice $\mathbb{L}$, there is a one-to-one correspondence between its tolerances and polarities. The correpondence is given by

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(iii) *There is a pair $f, g \colon L \longrightarrow L$ of order-preserving mappings such that* $\sigma_S = f \circ \sigma_T$ *and* $\mu_S = g \circ \mu_T$.

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It was proved by I. Chajda, and J. Nieminen that for any finite direct product $L = \prod_{i=1}^{n} L_i$ of lattices the isomorphism $\mathrm{Tol}(L) \cong \prod_{i=1}^{n} \mathrm{Tol}(L_i)$ holds. Moreover, we proved:

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Remark. We note that the poset $(\mathrm{Tol}(L), \sqsubseteq)$ is not a lattice in general!

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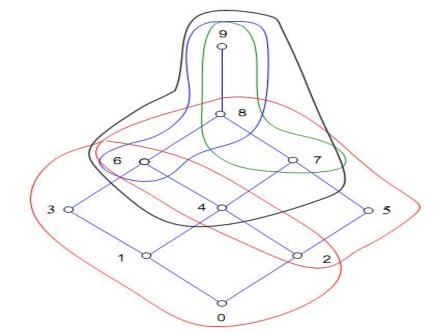

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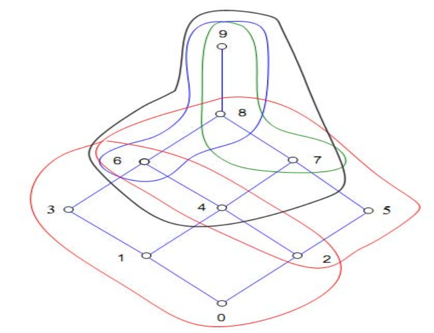

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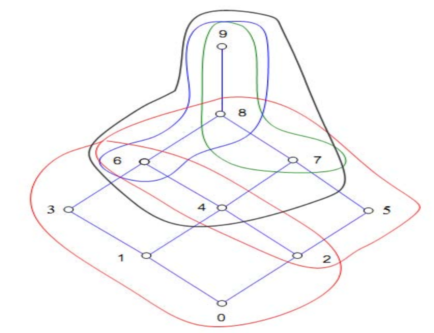

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Clearly, $T_1, T_2 \sqsubseteq S_1, S_2$ and $S_1 = T_1 \vee T_2$.

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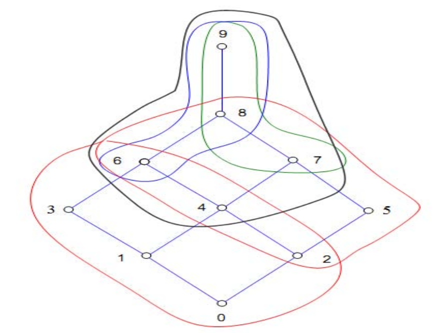

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Clearly, $T_1, T_2 \sqsubseteq S_1, S_2$ and $S_1 = T_1 \vee T_2$. If $\sup\{T_1, T_2\}$ would exist in $(\mathrm{Tol}(L), \sqsubseteq)$, then $S_1 = T_1 \vee T_2 \leq \sup\{T_1, T_2\} \leq S_1 \cap S_2 = S_1$ would imply $\sup\{T_1, T_2\} = S_1$, which is a contradiction, because S_2 is an upperbound for $\{T_1, T_2\}$, however $S_1 \sqsubseteq S_2$ does not hold.

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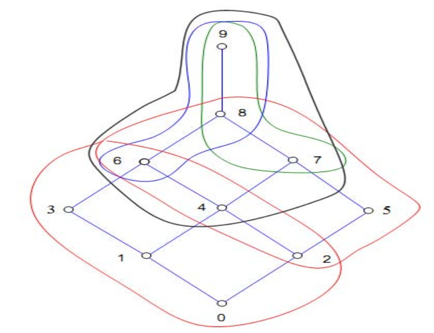

T_1: $\{0,1,2,4\}$, $\{1,3,4,6\}$, $\{2,4,5,7\}$, $\{4,6,7,8\}$, $\{6,8,9\}$;
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Lemma 2. *For any finite lattice L, $T \in \mathrm{Tol}(L)$ and any system $S_i \in \mathrm{Tol}(L)$, $i \in I$, $I \neq \emptyset$ of tolerances*

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(i) A *join-directoid* is an up-directed partially ordered set $\mathbb{A} = (A, \leq)$ where to any ordered pair $(x, y) \in A^2$ of elements a common upperbound $x \nabla y$ is assigned such that

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Remark. The choice of this common upperbound in general is not unique (provided x and y are not comparable). Hence an up-directed poset $(A, \leq)$ may be converted into several different join-directoids.

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Let $(A, \triangledown)$ be a join-directoid and $B \subseteq A$, $B \neq \emptyset$. $(B, \triangledown)$ is called a *subdirectoid* of $(A, \triangledown)$, whenever for any $b, c \in B$, $b \triangledown c \in B$ holds.

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(iii) *Let* $L = \prod_{i=1}^{n} L_i$, *and denote by* $\triangledown_i$ *the grupoid operation corresponding to the directoid* $(\mathrm{Tol}(L_i), \sqsubseteq_i)$. *Then* $(\mathrm{Tol}(L), \triangledown) \cong \prod_{i=1}^{n} (\mathrm{Tol}(L_i), \triangledown_i)$

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