On the tolerance lattice of tolerance factors
Part I

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Content

Tolerances

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All tolerances and all congruences of a lattice L form algebraic lattices with respect to $\subseteq$ denoted by $\mathrm{Tol}(L)$ and $\mathrm{Con}(L)$, respectively. $\mathrm{Con}(L)$ is not a sublattice of $\mathrm{Tol}(L)$, in general. (The meet operations are the same in both lattices, however the joins can be different.)

Factor lattices

A block of a tolerance $T \in \mathrm{Tol}(L)$ is such a maximal subset B of L that $B^2 \subseteq T$. Blocks are convex sublattices of L and it was shown by G. Czédli that they form a lattice (denoted by L/T) called the factor lattice of L modulo T.

Factor lattices modulo tolerances and modulo congruences

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For every $\varphi \in \mathrm{Con}(L))$, we have

$$\mathrm{Con}(L/\varphi) \cong [\varphi) \text{ (homomorphism theorem)}$$

Factor lattices modulo tolerances and modulo congruences

Moreover, for every $\varphi, \psi \in \mathrm{Con}(L)$ such that $\psi \geq \varphi$ we have a congruence ψ/φ induced on the factor lattice L/φ, such that

$$(L/\varphi)/(\psi/\varphi) \cong L/\psi \text{ (second isomorphism theorem)}.$$

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In this talk we formulate analogous results for factor lattices modulo tolerances.

Finite case

Since now on, all lattices are assumed to be finite.
In that case blocks of tolerances are intervals. Thus, if α is a block of $T \in \mathrm{Tol}(L)$, then we use the notation $\alpha = [0_\alpha, 1_\alpha]$.

Some properties of zeroes and units of blocks of tolerances

Let $a \in L$ and $T \in \mathrm{Tol}(L)$. We define

$$a_T = \bigwedge\{b \in L\colon (a,b) \in T\},$$
$$a^T = \bigvee\{b \in L\colon (a,b) \in T\}.$$

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$[a_T, (a_T)^T]$ and $[(a^T)_T, a^T]$ are blocks of T for any $a \in L$ and every block of T is of that form.
In other words, for any $\alpha \in L/T$ there are $b, c \in L$ such that $0_\alpha = b_T$ and $1_\alpha = c^T$.

Definition

Let L be a lattice, $T \in \mathrm{Tol}(L)$ and $\Theta \in \mathrm{Tol}(L/T)$.
We define a relation T^{Θ} on the lattice L in the following way:

$$(a,b) \in T^{\Theta} \iff \text{there are } \alpha, \beta \in L/T \text{ such that}$$

$$a \in \alpha,\ b \in \beta \text{ and } (\alpha, \beta) \in \Theta.$$

T^{Θ} is a tolerance

Proposition

For any $T \in \mathrm{Tol}(L)$ *and* $\Theta \in \mathrm{Tol}(L/T)$

(i) T^{Θ} *is a tolerance on* L*,*

(ii) $T \leq T^{\Theta}$,

(iii) *the mapping* $\phi_T\colon \mathrm{Tol}(L/T) \to [T)$ *given by* $\phi(\Theta) = T^{\Theta}$ *is order-preserving.*

We are going to call the relation T^{Θ} a *tolerance induced* on $[T)$ by Θ.

Blocks of T^{Θ}

Lemma

Let $T \in \mathrm{Tol}(L), \Theta \in \mathrm{Tol}(L/T), \alpha_1, \alpha_2 \in L/T$ *and* $\alpha_1 \leq \alpha_2$*. Then* $[\alpha_1, \alpha_2]$ *is a block of* Θ *if and only if* $[0_{\alpha_1}, 1_{\alpha_2}]$ *is a block of* T^{Θ}*.*

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Corollary

All blocks of T^{Θ} for $T \in \mathrm{Tol}(L)$ and $\Theta \in \mathrm{Tol}(L/T)$ are of the form $[0_{\alpha_1}, 1_{\alpha_2}]$, where $[\alpha_1, \alpha_2]$ is a block of Θ.

A new order on Tol(L)

Let $T, S \in \mathrm{Tol}(L)$. We say that T *fits into* S and write $T \sqsubseteq S$ iff $T \leq S$ and for any preblock X of T such that $X \subseteq \alpha$ for some $\alpha \in L/S$ there is a block $\beta \in L/T$ such that $X \subseteq \beta \subseteq \alpha$.

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Proposition

For any lattice L

(i) $\sqsubseteq$ *is a partial order on* $\mathrm{Tol}(L)$.

(ii) *If* $T \in \mathrm{Tol}(L), S \in \mathrm{Con}(L)$ *and* $T \leq S$, *then* $T \sqsubseteq S$.

The factor relation

Let $T, S \in \mathrm{Tol}(L)$ and $T \leq S$. We define a binary relation S/T on L/T in the following way:

$$(\beta_1, \beta_2) \in S/T \iff \beta_1, \beta_2 \subseteq \alpha \text{ for some } \alpha \in L/S.$$

We will call the above relation *the factor relation* of S modulo T.

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Proposition

Let $T, S \in \mathrm{Tol}(L)$. If $T \sqsubseteq S$, then the factor relation S/T is a tolerance on L/T. Moreover, $T^{S/T} = S$.

Some characterization of the relation $\sqsubseteq$

Proposition

For any lattice L and $T, S \in \mathrm{Tol}(L)$ such that $T \leq S$ the following conditions are equivalent:

(i) $T \sqsubseteq S$.

(ii) *For every $a \in L$*

$$((a_S)^T)_T = a_S \text{ and } ((a^S)_T)^T = a^S.$$

Some characterization of the relation $\sqsubseteq$

Corollary

Let $T, S \in \mathrm{Tol}(L)$ and $T \leq S$. Then $T \sqsubseteq S$ if and only if for any $a \in L$ there are $b, c \in L$ such that $a_S = b_T$ and $a^S = c^T$.

In other words, a tolerance T fits into S if and only if the zero of any block of S coincides with the zero of some block of T and the same applies to units.

Another characterization

Lemma

For any lattice L and $T, S \in \mathrm{Tol}(L)$ such that $T \leq S$ the following conditions are equivalent:

(i) $T \sqsubseteq S$.

(ii) *For any $a \in L$ and $\alpha \in L/S$ such that $a \in \alpha$ there exists $\beta \in L/T$ such that $a \in \beta \subseteq \alpha$.*

(iii) *Every block of S is the union of blocks of T included in it.*

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Lemma

Let $T, S \in \mathrm{Tol}(L)$ and $T \sqsubseteq S$. Then all blocks of the factor tolerance S/T are of the form

$$\chi_\alpha = \{\beta \in L/T \colon \beta \subseteq \alpha\}, \qquad \text{for } \alpha \in L/T.$$

The second isomorphism theorem

Theorem

For any $T, S \in \mathrm{Tol}(L)$ *such that* $T \sqsubseteq S$

$$(L/T)/(S/T) \cong L/S.$$

The relations defined before are strictly connected

Let $T \in \mathrm{Tol}(L)$. Now, we can see the connection between the relation induced on $[T)$ by $\Theta \in \mathrm{Tol}(L/T)$ and the factor tolerance.

Theorem

Let $T \in \mathrm{Tol}(L)$, $\Theta \in \mathrm{Tol}(L/T)$. Then

(i) $T \sqsubseteq T^{\Theta}$;

(ii) *all blocks of T^{Θ} are of the form $\alpha_\chi = \bigcup\{\beta \colon \beta \in \chi\}$, where χ is a block of Θ;*

(iii) $T^{\Theta}/T = \Theta$;

(iv) $L/T^{\Theta} \cong (L/T)/\Theta$.