Sectionally Complemented Lattices.
(A Mathematical Joke?)
Conference on
Universal Algebra and Lattice Theory
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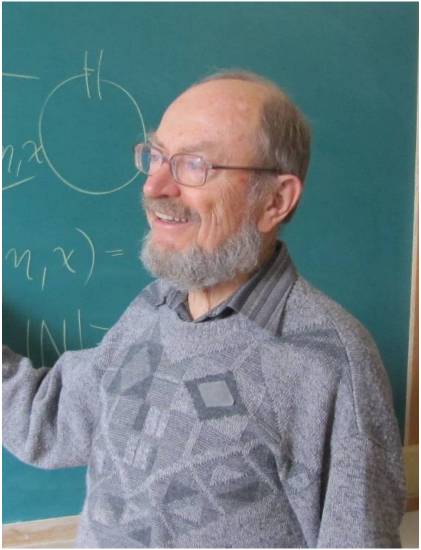
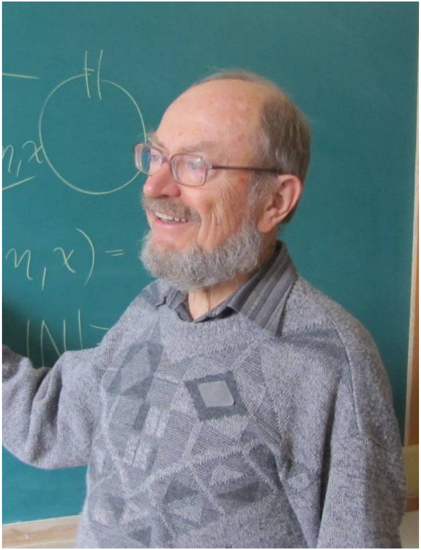
Salutation



- We celebrate Béla on his 80th birthday

Salutation



- We celebrate Béla on his 80th birthday
- Béla is up front; the circle is a group

A Joke

- What is polygamy?

A Joke

- What is polygamy?
- One wife too many.

A Joke

- What is polygamy?
- One wife too many.
- What is monogamy?

A Joke

- What is polygamy?
- One wife too many.
- What is monogamy?
- The same.

The Story

Half a century ago, I discovered with Schmidt the *1960 construction* of a chopped lattice M
(made up of sectionally complemented lattices)
and the *1960 sectional complement* $\mathbf{S}_{1960}$
(we'll describe these soon).
The idea was: for the ideals $I \subseteq J$ of the chopped lattice M,
form the sectional complements of I in J in each part, call this $\mathbf{S}$.
In general, $\mathbf{S}$ is not an ideal.
Throw away everything that could cause problems,
and show that what was left, $\mathbf{S}_{1960}$, is still enough.

In 2006, I started working with Roddy going in the opposite direction:

(i) Make a tiny correction (almost anywhere) in $\mathbf{S}$;

(ii) carry on, until no more corrections are left.

This algorithm diverges at every step.
We figured that we get a large set (the 2006 sectional complements)

$$\mathrm{SC}_{2006}(I, J)$$

of sectional complements of I in J.
We were interested in the size of $\mathrm{SC}_{2006}(I, J)$,
how big it is compared to $\mathrm{SC}(I, J)$,
the set of all sectional complements of I in J,
and whether $\mathbf{S}_{1960}$ belongs to it.

Conjectures

The set $\mathrm{SC}_{2006}(I, J)$ (the sectional complements of I in J obtained by the algorithm) is large, but small compared to $\mathrm{SC}(I, J)$ (the set of *all* sectional complements):

$$\frac{1}{100}|\mathrm{SC}(I, J)| \leq |\mathrm{SC}_{2006}(I, J)| \leq \frac{1}{16}|\mathrm{SC}(I, J)|.$$

We had no idea whether $\mathbf{S_{1960}}$ belongs to $\mathrm{SC}(I, J)_{2006}$ (in all examples, it did).

1940

So let me tell the story.
R. P. Dilworth proved the following result—it appeared as an exercise (with an asterisk) in Birkhoff's *Lattice Theory* (1948):

Theorem

Every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

1960

This is our result:

Theorem

Every finite distributive lattice D can be represented as the congruence lattice of a finite sectionally complemented *lattice L.*

A lattice L with 0 is *sectionally complemented* if all intervals $[0, a]$ are complemented, that is, for all $a \leq b$ in L, there is an element $c \in L$ satisfying $a \wedge c = 0$ and $a \vee c = 1$.

We rephrase our result as follows:

Theorem

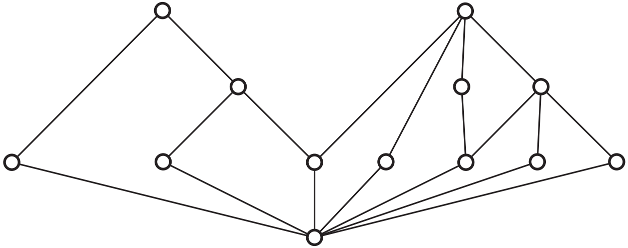
Let P be a finite order. Then there exists a sectionally complemented lattice L such that the order of join-irreducible congruences, $\mathrm{Con}_{\mathrm{Ji}}\, L$, is isomorphic to P.

Chopped Lattices

This is the main technical tool:

A *chopped lattice* is a finite meet-semilattice $(A; \wedge)$ regarded as a *partial algebra* $(A; \wedge, \vee)$, where $\vee$ is a partial operation defined as follows: $a \vee b$ is defined and $a \vee b = c$ iff c is the least upper bound of a and b for $a, b, c \in A$.

Here is an example:

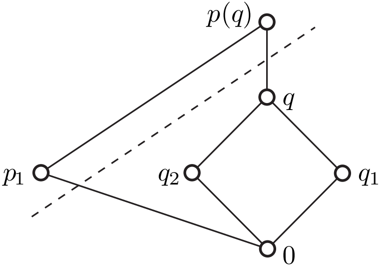


We denote by Max the set of maximal elements of A.

It is not difficult to see that the congruence lattice of a chopped lattice A is isomorphic to the congruence lattice of the ideal lattice of A, $\operatorname{Id} A$, so if we construct a *chopped lattice* A with the appropriate congruence lattice, the we also have a *lattice* with the same congruence lattice.

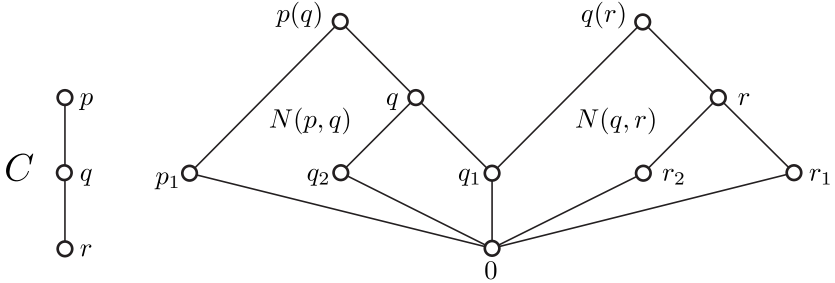
Constructing M

We construct the chopped lattice M for the Theorem from copies of the lattice $\mathsf{N}_6 = N(p, q)$, for $p \succ q \in P$, which has exactly one nontrivial congruence.



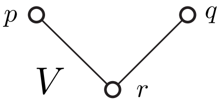
Example 1: P is the three-element chain C

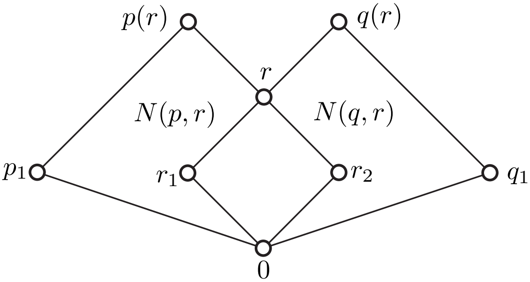
Let $C = \{p, q, r\}$ with $r \prec q \prec p$. We take two copies of the gadget, $N(p, q)$ and $N(q, r)$; they share the ideal $I = \{0, q_1\}$. So we can "merge" them and form the chopped lattice M_C:



Example 2: P is the three-element order V

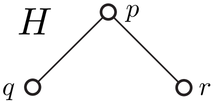
We take two copies of the gadget, $N(p,r)$ and $N(q,r)$; they share the ideal $J = \{0, r_1, r_2, r\}$; we "merge" them to form the chopped lattice M_V:

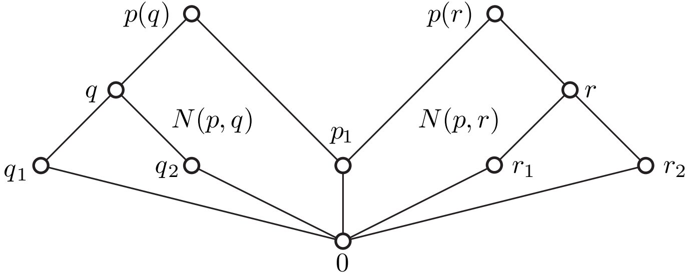




Example 3: P is the three-element order H

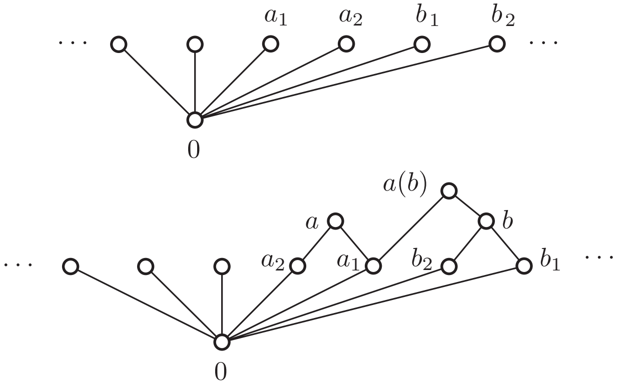
The chopped lattice M_H:





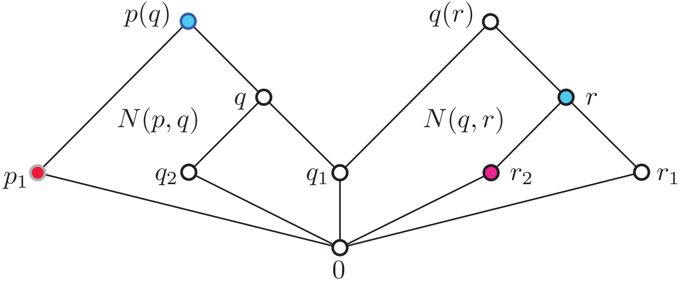
The general construction

Now it is easy to visualize the general construction of the chopped lattice M: instead of the three atoms as in the examples, we start with enough atoms to reflect the structure of P. Whenever $b \prec a$ in P, we build a copy of $N(a, b)$. It is routine to check that M has the required congruence lattice.

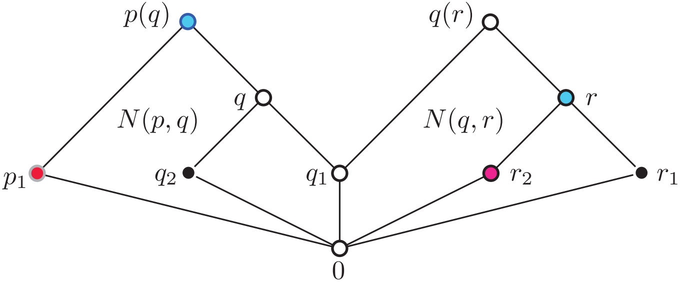


Is Id M sectionally complemented?

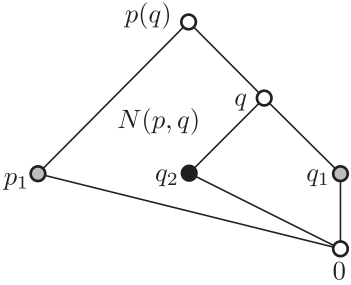
Let $I \subseteq J$ be ideals of M. On the diagrams, ideals will be marked by their "peaks", like this:



Choosing suitable sectional complements, we get the “black” ideal S, a sectional complement.



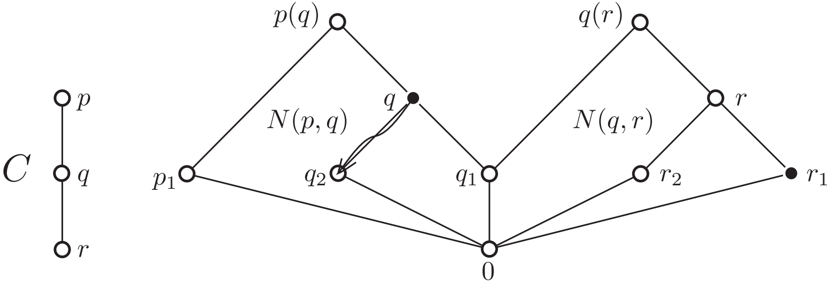
In general, if $I \subseteq J$, we can take the ideal S of M generated by all the atoms $a \in J - I$. The problem is, some atoms are trouble makers: if $p \succ q$ in P and $p_1, q_1 \in J - I$ (gray-filled), $q_2 \in I$ (black-filled), then q_2 is in the ideal generated by $J - I$, so it is not disjoint to I (they have q_2 in common).



The *1960 construction* of $\mathbf{S}_{1960}$: throw away *all the trouble makers* from $J - I$. Then $\mathbf{S}_{1960}$ is the ideal generated by the atoms we keep.

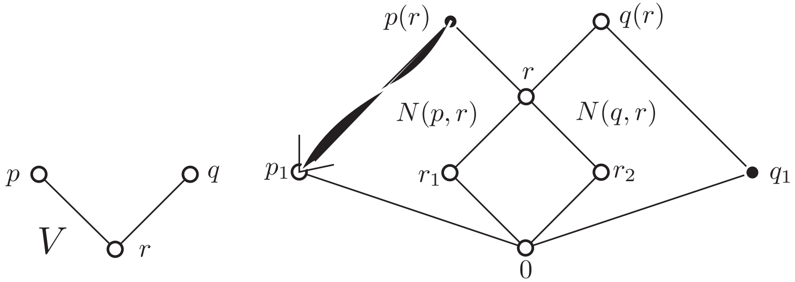
2006

If we get a candidate for an ideal (the black-filled elements in $N(p,q)$ and $N(q,r)$) that is not an ideal, it fails to be an ideal in a part that is a C, a V, or an H.
Here is a C example: q, r_1 (a C-failure) “cut to” q_2, r_1.



We call this a *C-cut.*
A C-failure at $p \succ q \succ r$ is *minimal* if there is no C-failure at $p' \succ q' \succ r'$ with $q' < q$.

Here is a V example: $p(r)$, q_1 (a V-failure) "cut to" p_1, q_1.



We call this a *V-cut.*

The Algorithm

Let $I \subseteq J$ be ideals of M. For each peak of I in a peak of J, choose a *maximal* sectional complement. Starting with these peaks:

Step 1. Look for a V-failure, and perform the corresponding V-cut, obtaining the new peaks.
Step 2. Repeat Step 1 until there are no more V-failures.
Step 3. Look for a *minimal* C-failure, and perform the corresponding C-cut.
Step 4. Repeat Step 3 until there are no more C-failures.

The Algorithm clearly terminates since the peaks are getting smaller and M is finite. Here is my main result with Roddy:

Theorem

When the Algorithm terminates, it finds a sectional complement. Hence the ideal lattice of M is sectionally complemented.

Note that a cut changes the peaks, so as a result of a cut another failure may disappear.

2010

With two students, Klus and Nguyen, we started working on the conjecture: $\mathrm{SC}_{2006}(I, J)$ (the sectional complements obtained by the algorithm) is large, but small compared to $\mathrm{SC}(I, J)$ (the set of all sectional complements).
We did not find what we expected.

The Punch Line

Theorem (Punch Line)

Let Σ be any sequence of cuts in the Algorithm. Then the sectional complement, $\mathbf{S}_\Sigma(I, J)$, is independent of Σ and

$$\mathbf{S}_\Sigma(I, J) = \{\mathbf{S}_{\mathbf{1960}}(I, J)\}.$$

Let Us Do Some Math

For a *chopped lattice* A, the congruence lattice of A is isomorphic to the congruence lattice of the ideal lattice, $\operatorname{Id} A$ (Grätzer-Lakser).

One step in the proof: let $\boldsymbol{\alpha}$ be a congruence relation of A.
For $I, J \in \operatorname{Id} A$, define the relation $\overline{\boldsymbol{\alpha}}$:

$$I \equiv J \pmod{\overline{\boldsymbol{\alpha}}} \quad \text{if} \quad I/\boldsymbol{\alpha} = J/\boldsymbol{\alpha}.$$

($I/\boldsymbol{\alpha}$ is the image of I in A/α.)
We claim that $\overline{\boldsymbol{\alpha}}$ is a congruence on $\operatorname{Id} A$.
To prove that if

$$I \equiv J \pmod{\overline{\boldsymbol{\alpha}}},$$

then

$$I \vee N \equiv J \vee N \pmod{\overline{\boldsymbol{\alpha}}},$$

we need a description of the join $A \vee B$ of two ideals A and B.

Define the set $U(A, B)_i \subseteq A$ inductively for all $0 < i < \omega$. Let

$$U(A, B)_0 = A \cup B.$$

If $U(A, B)_{i-1}$ is defined, then let $U(A, B)_i$ be the set of all $x \in A$ for which there are $u, v \in U(A, B)_{i-1}$ such that $u \vee v$ is defined in A and $x \leq u \vee v$. Then

$$A \vee B = \bigcup(\, U(A, B)_i \mid i < \omega\,).$$

The rest is easy.

Working with Peaks

For a chopped lattice A,

$$A = \bigcup (\,\mathrm{id}(m) \mid m \in \mathrm{Max}\,)$$

and each $\mathrm{id}(m)$ is a (finite) lattice.
(Recall that Max is the set of maximal elements of A.)
A *vector* (associated with A) is of the form $(i_m \mid m \in \mathrm{Max})$,
where $i_m \in \mathrm{id}(m)$ for all $m \in A$.
We order the vectors componentwise.
With every ideal I of A, we can associate the vector
$(i_m \mid m \in \mathrm{Max})$ defined by $I \cap \mathrm{id}(m) = \mathrm{id}(i_m)$ (the peaks). Clearly,
$I = \bigcup (\,\mathrm{id}(i_m) \mid m \in A\,)$. Such vectors are easy to characterize.

Let us call the vector $(j_m \mid m \in \mathrm{Max})$ *compatible* if $j_m \wedge n = j_n \wedge m$, for all $m, n \in \mathrm{Max}$.

Lemma

Let A be a chopped lattice.

(i) *There is a one-to-one correspondence between ideals and compatible vectors of A.*

(ii) *Given any vector $\mathbf{g} = (g_m \mid m \in \mathrm{Max})$, there is a smallest compatible vector $\overline{\mathbf{g}} = (i_m \mid m \in \mathrm{Max})$ containing $\mathbf{g}$.*

(iii) *Let I and J be ideals of A, with corresponding compatible vectors $(i_m \mid m \in \mathrm{Max})$ and $(j_m \mid m \in \mathrm{Max})$. Then*

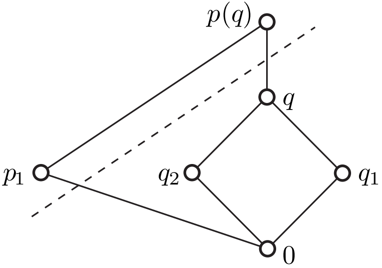
  (a) *$I \leq J$ in $\mathrm{Id}\, A$ iff $i_m \leq j_m$ for all $m \in \mathrm{Max}$.*

  (b) *The compatible vector corresponding to $I \wedge J$ is $(i_m \wedge j_m \mid m \in \mathrm{Max})$.*

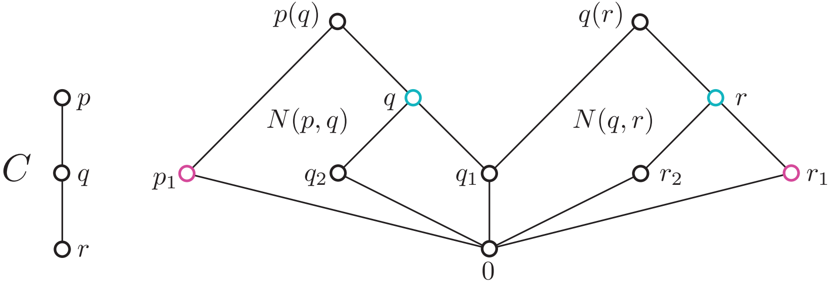
  (c) *Let $\mathbf{a} = (i_m \vee j_m \mid m \in \mathrm{Max})$. Then the $\overline{\mathbf{a}}$ is the compatible vector corresponding to $I \vee J$.*

Examples of M

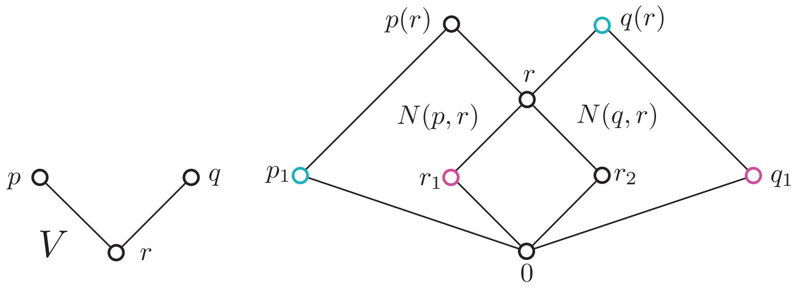
Recall that we construct the chopped lattice M for the Theorem from copies of the lattice $N_6 = N(p, q)$, for $p \succ q \in P$.



So $\mathrm{Max} = \{\, p(q) \mid p \succ q \in P \,\}$.



Red: compatible vector, $p_1 \wedge q(r) = p(r) \wedge r_1$
Green: not compatible vector, $q \wedge q(r) \neq p(q) \wedge r$

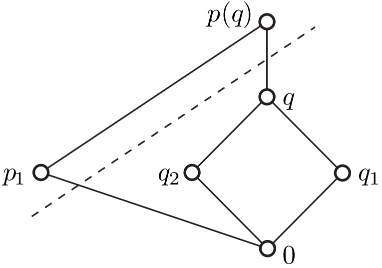


Red: not compatible vector, $r_1 \wedge q(r) = p(q) \wedge r_1$
Green: not compatible vector, $p_1 \wedge q(r) \neq p(r) \wedge q(r)$
Max $= (p(r), q(r))$ is a compatible vextor

Starting the Formal Definitions

Elements of M are compatible vectors:
$\mathbf{v} = (v_{p,q} \mid p \succ q \in P,\ v_{p,q} \in N(p,q))$.

We begin with some definitions utilizing the fact that in $N(p,q)$, for every $x \leq y$, there is a unique sectional complement z of x in y, except for $x = p_1$ and $y = p(q)$, in which case, there are three, q, q_1, and q_2; of these, q is maximal.



The Vector $\mathbf{s}$

Let $\mathbf{u}$ and $\mathbf{v}$ be compatible vectors of the chopped lattice M with $\mathbf{u} \leq \mathbf{v}$, that is, let $\mathbf{u} = (u_{xy} \mid x \succ y \in P)$ and $\mathbf{v} = (v_{xy} \mid x \succ y \in P)$, with $u_{xy} \leq v_{xy}$ in $N(x, y)$ for all $x \succ y \in P$.
Define the vector $\mathbf{s} = (s_{xy} \mid x \succ y \in P)$, where s_{xy} is the *maximal sectional complement* of u_{xy} in v_{xy}.

C-compatibility

Let $p \succ q \succ r$ in P, that is, let $\{p, q, r\}$ be a cover-preserving suborder C in P.

(i) We call a vector $\mathbf{c} = (c_{xy} \mid x \succ y \in P)$ *C-compatible* at $\{p, q, r\}$, if $c_{pq} \wedge q_1 = c_{qr} \wedge q_1$ in M. Otherwise, $\mathbf{c}$ is *C-incompatible* at $\{p, q, r\}$.

(ii) The vector $\mathbf{c}$ is *C-compatible*, if it has no C-incompatibilty.

(iii) We say that $\mathbf{c}$ has a *C-failure* at $\{p, q, r\}$, if $\mathbf{c}$ is C-incompatible at $\{p, q, r\}$ and, additionally, $c_{pr} = s_{pr}$ and $c_{qr} = s_{qr}$, that is, $\mathbf{c} = \mathbf{s}$ on $\{p, q, r\}$.

(iv) A $C(p, q, r)$-failure for $\mathbf{c}$ at $p \succ q \succ r$ is *minimal* if there is no $C(p', q', r')$-failure for $\mathbf{c}$ with $q' < q$.

Similarly, we define V*-compatibility* and H*-compatibility*.

A vector $\mathbf{c}$ *is compatible iff it is*
C*-compatible,*
V*-compatible, and*
H*-compatible.*

C-failure Lemma

Recall that $\mathbf{u} \leq \mathbf{v}$ are compatible vectors
and the vector $\mathbf{s}$ is constructed from $\mathbf{u}$ and $\mathbf{v}$
(maximal sectional complements).

Lemma

For a vector $\mathbf{c}$*, a* $C(p, q, r)$*-failure is represented by a row in the following table with* $c_{pq} = s_{pq}$ *and* $c_{qr} = s_{qr}$*:*

u_{pq}	u_{qr}	v_{pq}	v_{qr}	s_{pq}	s_{qr}
q_2	0	$p(q)$	p_1	p_1	q_1
q_2	0	$p(q)$	$q(r)$	p_1	$q(r)$
q_2	r_1	$p(q)$	$q(r)$	p_1	q_1
q_2	r_2	$p(q)$	$q(r)$	p_1	q_1
q_2	r	$p(q)$	$q(r)$	p_1	q_1

C-cuts

Let $\mathbf{c}$ be a vector with a $C(p,q,r)$-failure. The *C-cut* of $\mathbf{c}$ is a vector $R_C(\mathbf{c})$ all but one of whose components are the same as those of $\mathbf{c}$. One component of $\mathbf{c}$ is "cut" (substituted by an element it covers) as shown in this table (the C-cut Table):

u_{pq}	u_{qr}	v_{pq}	v_{qr}	$c_{pq} = s_{pq}$	$c_{qr} = s_{qr}$	
q_2	0	$p(q)$	q_1	p_1	q_1	$c_{qr} \mapsto 0$
q_2	0	$p(q)$	$q(r)$	p_1	$q(r)$	$c_{qr} \mapsto r$
q_2	r_1	$p(q)$	$q(r)$	p_1	q_1	$c_{qr} \mapsto 0$
q_2	r_2	$p(q)$	$q(r)$	p_1	q_1	$c_{qr} \mapsto 0$
q_2	r	$p(q)$	$q(r)$	p_1	q_1	$c_{qr} \mapsto 0$

V-failure and V-cuts

Similarly, we have the V-failure Lemma, V-cut Table, and V-cuts.

The Algorithm

Step 1. Set $\mathbf{c} = \mathbf{s}$.
Step 2. Look for a V-failure, and perform the corresponding V-cut, obtaining a new $\mathbf{c} = R_V(\mathbf{c})$.
Step 3. Repeat Step 2 until there are no more V-failures.
Step 4. Look for a *minimal* C-failure, and perform the corresponding C-cut, obtaining a new $\mathbf{c} = R_C(\mathbf{c})$.
Step 5. Repeat Step 4 until there are no more C-failures.

Since M is finite and $R_C(\mathbf{c}), R_V(\mathbf{c}) < \mathbf{c}$, the process must terminate, yielding a vector $\mathbf{s}^*$.

The $\mathbf{s}^*$ Theorem

Given the compatible vectors $\mathbf{u} \leq \mathbf{v}$, the vector $\mathbf{s}$, and a vector $\mathbf{s}^*$, a result of the Algorithm, we have the following result (Grätzer and Roddy):

Theorem

The vector $\mathbf{s}^*$ is compatible and it is a sectional complement of $\mathbf{u}$ in $\mathbf{v}$ in $\operatorname{Id} M$. Hence the lattice $\operatorname{Id} M$ is sectionally complemented. Moreover, for every $p \succ q$ in P, either $s^*_{pq} = s_{pq}$ or $s^*_{pq} \prec s_{p,q}$ holds.

Lament[1]

The proof is 15 pages long, comprising 41 cases, each from one line to half a page long. If the statement of any one of these cases fails, the theorem collapses.

[1] An expression of regret or disappointment

The Invariance for Step 3

The result with Klus and Nguyen (for any sequence of cuts, Σ, in the Algorithm, the sectional complement, $\mathbf{S}_{\Sigma}(I,J)$, is independent of Σ and $\mathbf{S}_{\Sigma}(I,J) = \{\mathbf{S}_{\mathbf{1960}}(I,J)\}$) follows from a sequence of lemmas.

Let $\mathbf{m}^2$ denote the following vector:

$$m^2_{pr} = \begin{cases} 0, & \text{if } m_{pr} = r_i \\ & \text{and there is a } V(p,q,r)\text{-failure for some } q \succ r; \\ m_{pr}, & \text{otherwise.} \end{cases}$$

Lemma

At the end of Step 3, we obtain the vector $\mathbf{m}^2$, *independent of the sequence of* V*-cuts performed.*

Lemma

The vector $\mathbf{m}^2$ *is* V*-compatible.*

The Invariance for Step 5

$p \succ q \succ r$ in P
Let $C(p, q, r)$ be a C-suborder $(r \prec q \prec p)$. We call $r \prec q$ the *stem* of C.

Lemma

Let $\mathbf{m}^2$ have a C-failure at $C(p, q, r)$. Then any C-suborder of P with the same stem, $r \prec q$, also has a C-failure. Moreover, all these failures are resolved by the same cut.

Lemma

Let C_1 and C_2 be two minimal C-failures that do not share a stem. Then, after a C-cut at C_1, the chain C_2 still has a C-failure.

Lemma

Let Σ be any sequence of C-cuts on $\mathbf{m}^2$ such that the vector $\mathbf{m}^2_\Sigma$ obtained by Σ has no C-failures. Then $\mathbf{m}^2_\Sigma$ does not depend on Σ.

So we proved that $\mathbf{s}_\Sigma$ does not depend on the choice of Σ. Let $\mathbf{s}$ denote this vector.
Therefore, $\mathrm{SC}_{2006}(I, J) = \{\mathbf{s}\}$ is a singleton.
It remains to prove that $\mathbf{s} = \mathbf{s_{1960}}$.

Lemma

Let $\mathbf{u} \leq \mathbf{v}$ be vectors in P. Let $\mathbf{c}$ be a vector obtained in a step of the algorithm and let $\mathrm{Cut}(\mathbf{c})$ be the vector obtained in next step of the algorithm. If $\mathbf{s_{1960}} \leq \mathbf{c}$, then $\mathbf{s_{1960}} \leq \mathrm{Cut}(\mathbf{c})$.

Lemma

Let $\mathbf{c}$ be a compatible vector for which $\mathbf{s_{1960}} \leq \mathbf{c} \leq \mathbf{m}$. Then $\mathbf{c} = \mathbf{s_{1960}}$.

Problems

Problem

The Intuitive Algorithm: find a failure, cut it.
Does it work?

Problem

Are there different algorithms that find other sectional complements?

More Problems

Problem

Can we apply the Algorithm to more general classes of sectionally complemented chopped lattices? (*For counter examples, see my paper with Lakser and Roddy.*)

Problem

Is there a proof not utilizing the $\mathbf{s}^*$ *Theorem that the only sectional complements found by the Algorithm is* $\mathbf{s}_{1960}$*?*

For lot more problems, see the papers:

Notes on sectionally complemented lattices. I.–V.