DEPENDENCE SPACES

Conference on Universal Algebra and Lattice Theory Szeged, Hungary

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21–25 June 2012

In Linear Algebra, Steinitz exchange Lemma states that:
if $a \notin span(A \cup \{b\})$, then $b \notin span(A \cup \{a\})$.
In particutar, if A is independent and $a \notin span(A)$, then:
$A \cup \{a\}$ is independent.

According to F. Gécseg, H. Jürgensen [6] the result which is usually referred to as the "Exchange Lemma", states that for transitive dependence, every independent set can be extended to form a basis. Our aim is to discuss some interplay between the discussed notion of [1]-[2] and [6]-[7]. We present another proof of the result of N.J.S. Hughes [1] on Steinitz' exchange theorem for infinite bases in connection with the notions of transitive dependence, independence and dimension as introduced in [7] and [8]. In the proof we assume Kuratowski-Zorn's Lemma, as a requirement pointed in [6]. In our opinion the proof is simpler as those of [1].

We use a modification of the the notation of [1]-[2] and [6]: $a, b, c, ..., x, y, z, ...$ (with or without suffices) to denote the elements of a space **S** and $A, B, C, ..., X, Y, Z, ...$ for subsets of the given **S**, $\mathbb{X}$, $\mathbb{Y}$,... denote a family of subsets of the space **S**, n is always a positive integer.

$A \cup B$ denotes the union of sets A and B, $A + B$ denotes the disjoint union of A and B, $A - B$ denotes the difference of A and B, i.e. is the set of those elements of A which are not in B.

The following definition is due to N.J.S. Hughes and was invented in 1962 in [1]:

Definition

A set **S** is called a *dependence space* if there is defined a set Δ, whose members are finite subsets of **S**, each containing at least 2 elements, and if the Transitivity Axiom is satisfied.

Definition

A set A is called *directly dependent* if $A \in \Delta$.

Definition

An element x is called *dependent on* A and is denoted by $x \sim \Sigma A$ if either $x \in A$ or if there exist distinct elements $x_0, x_1, ..., x_n$ such that

$$(1)\ \{x_0, x_1, ..., x_n\} \in \Delta$$

where $x_0 = x$ and $x_1, ..., x_n \in A$
and *directly dependent* on $\{x\}$ or $\{x_0, x_1, ..., x_n\}$, respectively.

Definition

A set A is called *dependent* if (1) is satisfied for some distinct elements $x_0, x_1, ..., x_n \in A$, and otherwise A is *independent*.

Definition

If a set A is *independent* and for any $x \in \mathbf{S}$, $x \sim \Sigma A$, i.e. x is dependent on A, then A is called a *basis of* $\mathbf{S}$.

Another words, a baisis is an independent set that span the whole space.

Definition

TRANSITIVITY AXIOM:

If $x \sim \Sigma A$ and for all $a \in A$, $a \sim \Sigma B$, then $x \sim \Sigma B$.

A similar definition of a *dependence* D was introduced in [6]-[7]. First we recall the definition of [6], p. 425 and [7]:

Definition

Let $\mathbf{S}$ be a set with dependence D and $a \in \mathbf{S}$, $A \subseteq \mathbf{S}$. The element x is said to *depend on* A if $a \in A$ or there is an independent subset $A' \subseteq A$ such that $A' \cup \{a\}$ is dependent in $\mathbf{S}$.

Lemma

In a dependence space **S** *the following are equivalent for an element* $x \in \mathbf{S}$ *and a subset* A *of* **S**:
(i) $x \sim \Sigma A$ *in the sense of [1] and*
(ii) x *depends of* A *in the sense of [6] and [7].*

Proof

Assume (i), i.e. that $x \sim \Sigma A$ in the sense of [1]. Therefore, $x \in A$ or there exists elements $x_0 = x$ and $x_1, ..., x_n \in A$ such that the condition (1) is satisfied, i.e.: $\{x_0, x_1, ..., x_n\} \in \Delta$. In the second case, let the set $A' = \{x_1, ..., x_n\} \subset A$ be a minimal set such that $\{x_0, x_1, ..., x_n\}$ is dependent in **S** in the sense of [1]. Then the set say A' is independent. If we assume that (ii) is not satisfied, i.e. for all independent subsets A'' of A, the set $\{x\} \cup A''$ is independent in **S**, then A'' is independent, for all $A'' \subseteq A$, a contradiction for $A' = A''$. We conclude (ii).

Assume now (ii), i.e. that x depends of A in the sense of [6] and [7]. Thus $x \in A$ or there exists an independent subset $A' \subseteq A$ such that $A' \cup \{x\}$ is dependent in **S**. In the second case we get that $\{x\} \cup A'$ is dependent in **S**, i.e. there exists $x_0 = x$ and $x_1, ..., x_n \in A'$, such that $x_0, x_1, ..., x_n \in \Delta$, as A' is independent. We conclude (i). $\square$

Following ideas of [6]-[7] we accept the following:

Definition

The *span* $< X >$ of a subset X of **S** is the set of all elements of **S** which depends on X, i.e. $x \in < X >$ iff $x \sim \Sigma X$. A subset X of **S** is called *closed* if $X = < X >$, and a dependence D is called *transitive*, if $< X > = << X >>$, for all subsets X of **S**.

First we note, that in a dependence space D, the notion of *transitive dependence* of [7] and [6] is equivalent to those of [1].

Theorem

Given a dependence space **S** *satisfying the Transitivity Axiom in the sense of [1]. Then dependence space* **S** *is transitive in the sense of [7] and [6]. And vice versa, if a dependence space* **S** *is transitive in the sense of [6]-[7], then the Transitivity Axiom of [1] is satisfied.*

Proof

Let a transitive axiom of [1] be satisfied in **S** and let $X \subseteq \mathbf{S}$. $X \subseteq < X >$ and therefore $< X > \subseteq << X >>$, by Remark 3.6 of [7]. Now let $x \in << X >>$, i.e. $x \sim \Sigma < X >$. But $y \sim \Sigma X$, for all $y \in < X >$. Thus by the Transitivity Axiom $x \sim \Sigma X$, i.e. $x \in < X >$. We get that $< X > = << X >>$.

Now, assume that a dependence is transitive in the sense of [6]-[7], i.e. **S** be a transitive dependence space, i.e. $<< X >>=< X >$ for all $X \subseteq \mathbf{S}$. Let $x \sim \Sigma A$ and for all $a \in A$, $a \sim \Sigma B$. Thus $x \in A$ or the set $\{x\} \cup A$ is dependent, i.e. $x \in< A >$. Moreover, $a \in< B >$, for all $a \in A$. Therefore $\{a\} \subseteq< B >$, i.e. $< a >\subseteq<< B >>=< B >$, by the Remark 3.6 of [6]. In consequence:
$A = \bigcup\{a : a \in A\} \subseteq \bigcup\{< a >: a \in A\} \subseteq< B >$ and thus:
$< A >\subseteq<< B >>=< B >$. We get that $x \in< B >$, i.e. $x \sim \Sigma B$, i.e. the Transitivity Axiom is satisfied. $\square$

Note, that the following well known properties (see [6]-[8]) are satisfied in a dependence space:

Proposition

(2) Any subset of an independent set A is independent.
(3) A basis is a maximal independent set of **S** *and vice versa.*
(4) A basis is a minimal subset of **S** *which spans* **S** *and vice versa.*
(5) The family $(\mathbb{X}, \subseteq)$ of all independent subsets of **S** *is partially ordered by the set-theoretical inclusion. Shortly we say that $\mathbb{X}$ is an ordered set (a po-set).*
(6) Any superset of a dependent set of **S** *is dependent.*

Lemma

In a dependence space, if $a \notin < A \cup \{b\} >$, then $b \notin < A \cup \{a\} >$.

Proof

If $b \in < A \cup \{a\} > - < A >$, then there exists $a_1, ..., a_n \in A$, such that $b \sim \{a, a_1, ..., a_n\}$, i.e. $\{a, a_1, ...a_n, b\} \in \Delta$.
Therefore $a \in < \{b\} \cup A >$. $\square$
In particutar, if A is independent and $a \notin < A >$, then:
$A \cup \{a\}$ is independent.

Following K. Kuratowski and A. Mostowski [4] p. 241, a po-set $(\mathbb{X}, \subseteq)$ is called *closed* if for every chain of sets $\mathbb{A} \subseteq P(\mathbb{X})$ there exists $\cup\mathbb{A}$ in $\mathbb{X}$, i.e. $\mathbb{A}$ has the supremum in $(\mathbb{X}, \subseteq)$.

Theorem

The po-set $(\mathbb{X}, \subseteq)$ *of all independent subsets of* $\mathbf{S}$ *is closed.*

Proof

Let $\mathbb{A}$ be a chain of independent subsets of **S**, i.e. $\mathbb{A} \subseteq P(\mathbb{X})$, and for all $A, B \in \mathbb{A}$ $(A \subseteq B)$ or $(B \subseteq A)$. We show that the set $\cup\mathbb{A}$ is independent. Otherwise there exist elements $\{x_0, x_1, ..., x_n\} \in \Delta$ such that $x_i \in \cup\mathbb{A}$, for $i = 0, ..., n$. Therefore there exists a set $A \in \mathbb{A}$ such that $x_i \in A$ for all $i = 0, ..., n$.
We conclude that A is dependent, $A \in \mathbb{A}$, a contradiction. $\square$

Theorem

The po-set $(\mathbb{C}(\mathbb{X}), \subseteq)$ *of all independent subsets of* **S** *is an algebraic closure system.*

Proof

Let $\mathbb{A}$ be a directed family of independent subsets of **S**, i.e. $\mathbb{A} \subseteq P(\mathbb{X})$, and for all $A, B \in \mathbb{A}$ there exists a $C \in \mathbb{A}$ such that $(A \subseteq C)$ and $(B \subseteq C)$. We show that the set $\cup\mathbb{A}$ is independent. Otherwise there exist elements $\{x_0, x_1, ..., x_n\} \in \Delta$ such that $x_i \in \cup\mathbb{A}$, for $i = 0, ..., n$. Therefore there exists a set $C \in \mathbb{A}$ such that $x_i \in C$ for all $i = 0, ..., n$.

We conclude that C is dependent, $C \in \mathbb{A}$, a contradiction. $\square$

A transfinite version of the Steinitz Exchange Theorem, provides that any independent subset injects into any generating subset. For more information on the role of Steinitz papers consult the book chapter *400 Jahre Moderne Algebra*, of [5]. The following is a generalization of Steinitz' Theorem originally proved in 1913 and then in [1]-[2]:

Theorem

If A is a basis and B is an independent subset (of a dependence space $\mathbf{S}$). Then assuming Kuratowski-Zorn Maximum Principle, there is a definite subset A' of A such that the set $B + (A - A')$ is also a basis of $\mathbf{S}$.

Proof
If B is a basis then B is a maximal independent subset of **S** and $A' = A$ is clear.
Assume that A is a basis and B is an independent subset (of the dependence space **S**). Consider $\mathbb{X}$ to be the family of all independent subsets of **S** containing B and contained in $A \cup B$. then $(\mathbb{X}, \subseteq)$ is well ordered and closed. Therefore assuming Kuratowski-Zorn Maximal Principle [3] there exists a maximal element of $\mathbb{X}$.
We show that this maximal element $X \in \mathbb{X}$ is a basis of **S**.

As $X \in \mathbb{X}$ then $B \subseteq X \subseteq A \cup B$ by the construction. Therefore $X = B + (A - A')$ for some $A' \subseteq A$. We show first that for all $a \in A$, $a \sim \Sigma X$. If $a \in X$ then $a \sim \Sigma X$ by the definition. If not, then put $Y = X + \{a\}$. Then $X \neq Y$, $X \subseteq Y$, $B \subseteq Y \subseteq A \cup B$ and Y is dependent in $\mathbf{S}$.

By the definition there exist elements: $\{x_0, x_1, ..., x_n\} \in \Delta$ with $x_1, ..., x_n \in X + \{a\}$, as X is an independent set. Moreover, one of x_i is a, say $x_0 = a$. We get $a \sim \Sigma X$ as $x_0, x_1, ..., x_n$ are different. Now we show that X is a basis of $\mathbf{S}$. Let $x \in S$, then $x \sim \Sigma A$ as A is a basis of $\mathbf{S}$. Moreover for all $a \in A$, $a \sim \Sigma X$, thus $x \sim \Sigma X$ by the Transitivity Axiom. $\square$

Theorem

If in a dependence space **S** *with a dependence D, such that D is transitive in the sense of [6] on all independent sets, then the transitivity axiom of [1] holds.*

Proof

Let $x \sim \Sigma A$ and $a \sim \Sigma B$ for all $a \in A$. Assume that x does not depend of B, i.e. the set $\{x\} \cup B$ is independent. Therefore B is independent. But $x \in< A >$ and $a \in< B >$, for all $a \in A$. Thus $\{a\} \subseteq< B >$ and consequently $< a >\subseteq<< B >>=< B >$, for all $a \in A$. Therefore $A \subseteq \bigcup\{< a >: a \in A\} \subseteq< B >$, and we get $< A >\subseteq<< B >>=< B >$. Finally, $x \in< B >$. $\square$

Remark

The example presented in [6] on p. 425 shows that the assumption on transitivity of a dependence space **S** on independent sets is an essential condition for the "Exchange Lemma". Namely, defining $S = \{a, b, c\}$ and $D = \{a, b\}, \{b, c\}, \{a, b, c\}$ one gets: $\emptyset, \{a\}, \{b\}, \{c\}$ and $\{a, c\}$ are independent sets. Moreover: $< a >= \{a, b\}$ and $<< a >>= \{a, b, c\}$. Therefore the dependence D is not transitive on independent sets. The bases are: $\{b\}$ and $\{a, c\}$. The "Exchange lemma" does not hold (for the base $A = \{b\}$ and the independent set $B = \{a\}$).

Remark

Let R denotes the set of real numbers. D is the set of all infinite subsets of R. Then a dependence D in the sense [6] is given, which is not a dependence space **S** in in the sense of [1].

[1] Norris C., Texas R. : *How to count to infinity?*, The Book of Ranger, 1971.
[2] Bauer F.L., *Historische Notizen zur Informatik*, Springer-Verlag Berlin Heidlberg 2009.
[3] Cohn P.M., *Universal Algebra*, Harper and Row, New York, 1965. Revised edition, D. Reidel Publishing Co., Dordrecht, 1981.
[4] Gécseg F., Jürgensen H., *Algebras with dimension*, Algebra Universalis, **30** (1993) 422–446.
[5] Hughes N.J.S., *Steinitz' Exchange Theorem for Infinite Bases*, Compositio Mathematica, tome 15, 1962–1964, p. 113–118.
[6] Hughes N.J.S., *Steinitz' Exchange Theorem for Infinite Bases II*, Compositio Mathematica, tome 17, 1965-1966, p. 152–155.

[7] Kuratowski K., Une mthode d'élimination des nombres transfinis des raisonnements mathématiques, Fund. Math. 3 (1922), p. 89.
[8] Kuratowski K., Mostowski A., *Teoria Mnogości*, PWN, Warszawa 1966.
[9] Welsh, D.J.A., *Matroid Theory*, Academic Press, London, 1976.

DZIĘKUJĘ! DANKE! KÖSZÖNÖM SZÉPEN, THANK YOU!