## Tolerances as congruence images Conference on Universal Algebra and Lattice Theory Dedicated to the 80-th birthday of Béla Csákány

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Szeged, Hungary, June 21-25, 2012

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In particular, the image of every congruence is a tolerance.

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Characterize all varieties in which every tolerance

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The variety of all lattices has TImC (Czédli, Grätzer).

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Theorem (Chajda, Czédli, Halaš, Lipparini)

Every variety defined by linear identities has TImC.

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• All algebras of a given similarity type.

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## A Mal'tsev-like condition

### Condition M(n)

For any pair (f,g) of 2*n*-ary terms such that the identity  $f(x_0, x_0, \dots, x_{n-1}, x_{n-1}) \approx g(x_0, x_0, \dots, x_{n-1}, x_{n-1})$  holds in  $\mathcal{V}$ ,

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#### Theorem (Czédli, Kiss)

A variety satisfies TImC iff it satisfies M(n) for every  $n \ge 1$ .

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#### Theorem (Czédli, Kiss)

A variety satisfies TImC iff it satisfies M(n) for every  $n \ge 1$ .

Remark: No finite set of conditions M(n) suffices.

Corollary

Every variety of lattices has TImC.



## Corollary

Every variety of lattices has TImC.

## Proof

If 
$$f(..., x, x, ...) \approx g(..., x, x, ...)$$
 is a lattice identity, then let  

$$h(..., x, y, u, v, ...) =$$

$$= f(..., x \land u, y \land v, ...) \lor g(..., y \land u, x \land v, ...).$$

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$$\begin{aligned} h(\ldots, \mathbf{x}, \mathbf{y}, \mathbf{x}, \mathbf{y}, \ldots) &= \\ &= f(\ldots, \mathbf{x} \land \mathbf{x}, \mathbf{y} \land \mathbf{y}, \ldots) \lor g(\ldots, \mathbf{y} \land \mathbf{x}, \mathbf{x} \land \mathbf{y}, \ldots), \\ \text{which is } f(\ldots, x, y, \ldots), \text{ since } f \text{ is monotone, and similarly,} \\ h(\ldots, \mathbf{y}, \mathbf{x}, \mathbf{x}, \mathbf{y}, \ldots) &= \\ &= f(\ldots, \mathbf{y} \land \mathbf{x}, \mathbf{x} \land \mathbf{y}, \ldots) \lor g(\ldots, \mathbf{x} \land \mathbf{x}, \mathbf{y} \land \mathbf{y}, \ldots), \\ \text{which is } g(\ldots, x, y, \ldots). \end{aligned}$$

### Corollary

Every variety of lattices has TImC.

#### Proof

If 
$$f(..., x, x, ...) \approx g(..., x, x, ...)$$
 is a lattice identity, then let  

$$h(..., \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, ...) =$$

$$= f(..., \mathbf{x} \wedge \mathbf{u}, \mathbf{y} \wedge \mathbf{v}, ...) \lor g(..., \mathbf{y} \wedge \mathbf{u}, \mathbf{x} \wedge \mathbf{v}, ...).$$

Then we have

 $h(\dots, \mathbf{x}, \mathbf{y}, \mathbf{x}, \mathbf{y}, \dots) =$   $= f(\dots, \mathbf{x} \land \mathbf{x}, \mathbf{y} \land \mathbf{y}, \dots) \lor g(\dots, \mathbf{y} \land \mathbf{x}, \mathbf{x} \land \mathbf{y}, \dots),$ which is  $f(\dots, x, y, \dots)$ , since f is monotone, and similarly,  $h(\dots, \mathbf{y}, \mathbf{x}, \mathbf{x}, \mathbf{y}, \dots) =$   $= f(\dots, \mathbf{y} \land \mathbf{x}, \mathbf{x} \land \mathbf{y}, \dots) \lor g(\dots, \mathbf{x} \land \mathbf{x}, \mathbf{y} \land \mathbf{y}, \dots),$ which is  $g(\dots, \mathbf{x}, \mathbf{y}, \dots)$ . Thus M(n) holds.

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## Further examples

### Positive results

The following varieties satisfy M(n) for all n (so have TImC).

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The following varieties satisfy M(n) for all n (so have TImC).

• The variety of semilattices.

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The following varieties satisfy M(n) for all n (so have TIMC).

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Lattices are idempotent algebras: t(x, x, ..., x) = x for every term;

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Lattices are idempotent algebras: t(x, x, ..., x) = x for every term; have a majority term:  $m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x$ .

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#### Example

There exists an idempotent variety with a majority term which fails TImC.

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There exists an idempotent variety with a majority term (generated by a 3-element algebra), which fails TImC.

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There exists an idempotent variety with a majority term (generated by a 3-element algebra), which fails TImC.

Rules out possible generalizations.

## Other varieties without TImC

#### Theorem

If a congruence *n*-permutable variety has TImC,



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## Other varieties without TImC

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## Other varieties without TImC

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**Proof**: by applying M(n) to the Mal'tsev condition discovered by Hagemann and Mitschke.

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### Preprint

Czédli, Kiss: Varieties whose tolerances are homomorphic images of their congruences, http://arxiv.org/pdf/1204.2228.pdf.