On the Lattice of Clones of Incompletely Specified Operations

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1. What is IS operation?
2. Lattice of IS clones
   - Some properties
   - A maximal IS clone
3. Lattice of IS clones on $A = \{0, 1\}$.
   - Cardinality of the lattice
   - Minimal IS clones on $A = \{0, 1\}$
   - Maximal IS clones on $A = \{0, 1\}$
4. IS operations preserving relations
5. IS operations via a one-point extension
   - Extended IS operations
   - Algebra of extended IS operations
   - Composition closed set of extended IS operations
   - Extended IS operations preserving relations

What is IS operation?

Total operation:

OR	0	1
0	0	1
1	1	1

Let

$$h(x_1, x_2) = \mathtt{OR}(g(x_1), x_2)$$

Partial operation:

$$\mathtt{OR}(g(x_1), 1) \text{ undefined if } g(x_1) \text{ is undefined}$$

Incompletely specified operation:

$$\mathtt{OR}(g(x_1), 1) \quad = \quad 1$$

How to define it formaly?

Let A be a finite set and $k \notin A$.
Partial operation:

$$f : A^n \to A \cup \{k\}, \qquad k - \text{undefined}$$

Incompletely specified operation:

$$f : A^n \to A \cup \{k\}, \qquad k - \text{unspecified}$$

I_A - set of all IS operations on A

New composition

Definition

Let $f \in I_A^{(n)}$ and $g_1, \ldots, g_n \in I_A^{(m)}$. The i-composition of f and $g_1, \ldots, g_n$ is an m-ary IS operation defined by

$$f[g_1, \ldots, g_n](x_1, \ldots, x_m) = \prod_{\substack{(y_1, \ldots, y_n) \in A^n, \\ y_i \sqsubseteq g_i(x_1, \ldots, x_m) \\ 1 \leq i \leq n}} f(y_1, \ldots, y_n)$$

where

$$\prod\{x_i : 1 \leq i \leq l\} = \begin{cases} x_1 & \text{, if } x_1 = x_2 = \ldots = x_l, \\ k & \text{, otherwise.} \end{cases}$$

$$\sqsubseteq = \{(x, x) : x \in A \cup \{k\}\} \cup \{(x, k) : x \in A\}$$

Example

$A = \{0, 1\}$

composition of partial operations

OR	0	1
0	0	1
1	1	1

	g_1	g_2
0	1	2
1	0	0

	$\mathrm{OR}(g_1, g_2)$
0	2
1	0

$\mathrm{OR}(g_1, g_2)(0) = \mathrm{OR}(g_1(0), g_2(0)) = 2$

i-composition of IS operations

OR	0	1
0	0	1
1	1	1

	g_1	g_2
0	1	2
1	0	0

	$\mathrm{OR}[g_1, g_2]$
0	1
1	0

$\mathrm{OR}[g_1, g_2](0) = \mathrm{OR}(g_1(0), g_2(0)) = \mathrm{OR}(1, 0) \sqcap \mathrm{OR}(1, 1) = 1 \sqcap 1 = 1$

IS clone

Definition

A set $C \subseteq I_A$ is called a clone of incompletely specified operations (or IS clone) if

- *C contains all projections and*
- *C is closed with respect to i-composition.*

IS clone

- for $f \in I_A^{(1)}$ let $\zeta f = \tau f = \Delta f = f$;
- for $f \in I_A^{(n)}, n \geq 2$, let $\zeta f, \tau f \in I_A^{(n)}$ and $\Delta f \in I_A^{(n-1)}$ be defined as
  - $(\zeta f)(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$
  - $(\tau f)(x_1, x_2, x_3, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n)$
  - $(\Delta f)(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_1, x_2 \ldots, x_{n-1})$
- for $f \in I_A^{(n)}$ and $g \in I_A^{(m)}$ let $f \diamond g \in I_A^{(m+n-1)}$ be defined as

$$(f \diamond g)(x_1, \ldots, x_{m+n-1}) = \prod_{\substack{y \in A \\ y \sqsubseteq g(x_1, \ldots, x_m)}} f(y, x_{m+1}, \ldots, x_{m+n-1})$$

Example

$A = \{0, 1\}$

OR	0	1
0	0	1
1	1	1

	g
0	0
1	2

Let $h(x_1, x_2) = \mathrm{OR}(g(x_1), x_2)$.

For partial operations:

h	0	1
0	0	1
1	2	2

$h(1, 1) = \mathrm{OR}(g(1), 1) = 2$

For IS operations:

h	0	1
0	0	1
1	2	1

$h(1, 1) = \mathrm{OR}(g(1), 1) = \mathrm{OR}(0, 1) \sqcap \mathrm{OR}(1, 1) = 1 \sqcap 1 = 1$

IS clone

$\mathcal{I}_A = (I_A; \diamond, \zeta, \tau, \Delta, e_1^{2,A})$ full algebra of IS operations

Theorem

$C \subseteq I_A$ is an IS clone if and only if C is a subuniverse of the full algebra of IS operations.

Some properties

$\mathcal{L}^i_A = (L^i_A, \subseteq)$, L^i_A - set of all IS clones on A.

- J_A is the least IS clone.
- I_A is the greatest IS clone.
- Intersection of IS clones is an IS clone.
- $\langle F \rangle_i = \cap\{C : C \text{ is an IS clone and } F \subseteq C\}$
  $\Rightarrow \langle\ \rangle_i : P(I_A) \to P(I_A)$ is an algebraic closure operator.
- $\mathcal{L}^i_A$ is an algebraic lattice.

A maximal IS clone

Theorem (Haddad, Rosenberg, Schweigert 1990)

$O_A \cup \langle\{c_k\}\rangle_p$ is a maximal partial clone on A.

Theorem

O_A is a maximal IS clone.

Cardinality of the lattice on $A = \{0, 1\}$

Lattice of IS clones is isomorphic to the lattice of hyperclones on $A = \{0, 1\}$.

$H_A \to I_A : \; f \mapsto f^{is}$

$$f^{is}(x_1, \ldots, x_n) = \begin{cases} 0 & , f(x_1, \ldots, x_n) = \{0\} \\ 1 & , f(x_1, \ldots, x_n) = \{1\} \\ 2 & , f(x_1, \ldots, x_n) = \{0, 1\} \end{cases}$$

Theorem (Machida 2002)

There are continuum many IS clones on $A = \{0, 1\}$.

Minimal IS clones on $A = \{0, 1\}$

Theorem (Post 1941)

There are 7 minimal clones on $A = \{0, 1\}$.

Theorem (Börner, Haddad, Pöschel 1991)

There are 11 minimal partial clones on $A = \{0, 1\}$.

Theorem (Pantović, Vojvodić 2004)

There are 13 minimal IS clones on $A = \{0, 1\}$.

m_{12}	0	1
0	0	2
1	1	1

m_{13}	0	1
0	0	0
1	2	1

Maximal IS clones on $A = \{0, 1\}$

Theorem (Post 1941)

There are 5 maximal clones on $A = \{0, 1\}$.

Theorem (Freivald 1966)

There are 8 maximal partial clones on $A = \{0, 1\}$.

Theorem (Tarasov 1974)

There are 9 maximal IS clones on $A = \{0, 1\}$.

$$\begin{aligned} M_1 &= iPol(0 \;\; 1) \\ M_9 &= iPol\left(\{0,1,2\}^3 \setminus \{(0,1,1),(1,0,0)\}\right) \end{aligned}$$

Weak extension of ρ

Let $\rho \subseteq A^m$. The weak extension of ρ is the relation ρ_w defined by

$$\begin{aligned}\rho_w \quad = \quad & \{(a_1, \ldots, a_m) \in (A \cup \{k\})^m : \text{there is } (b_1, \ldots, b_m) \text{ such that} \\ & (b_1, \ldots, b_m) \in \rho \text{ and } (b_1, \ldots, b_m) \sqsubseteq (a_1, \ldots, a_m)\}.\end{aligned}$$

Definition

$f \in I_A^{(n)}$ u-preserves ρ iff for all $A_1, \ldots, A_m \in A^n$:

$$\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} \subseteq \rho \Rightarrow \begin{pmatrix} f(A_1) \\ \vdots \\ f(A_m) \end{pmatrix} \in \rho_w$$

Example (weak extension)

$A = \{0, 1\}$

$$\rho = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho_w = \begin{pmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}$$

$$\rho' = \begin{pmatrix} 0 & 0 & 0 & 2 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 & 2 & 1 & 2 \end{pmatrix}$$

(ρ' is not the weak extension of ρ)

Weak extension of ρ

Theorem

Let $m \geq 1$ and $\rho \subseteq A^m$. Then $uPol\rho$ is an IS clone.

Theorem

If g u-preserves ρ and $g \sqsubseteq f$ then f u-preserves ρ.

One-point extension

Let us define the mapping $I_A \to O_{A\cup\{k\}} : \; f \mapsto f^+$, as follows:

$$f^+(x_1,\ldots,x_n) = \prod_{\substack{(y_1,\ldots,y_n)\in A^n,\\ (y_1,\ldots,y_n)\sqsubseteq(x_1,\ \ldots,\ x_m)}} f(y_1,\ldots,y_n)$$

Example (one-point extension)

$A = \{0, 1\}$
Partial operation:

OR^+	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\mathrm{OR}^+(2, 1) = 2$
Incompletely specified operation:

OR^+	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$\mathrm{OR}^+(2, 1) = \mathrm{OR}(0, 1) \sqcap \mathrm{OR}(1, 1) = 1 \sqcap 1 = 1$

Some denotations

f	0	1
0	0	1
1	1	1

$\Rightarrow$

f^+	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

g	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$\Rightarrow$

g^-	0	1
0	0	1
1	1	1

$$F^+ = \{f^+ : f \in F\} \subseteq O_{A\cup\{k\}} \quad \text{for } F \subseteq I_A.$$

F^+ is the extension of F

Algebra of extended IS operations

Full algebra of operations on $A \cup \{k\}$:

$$\mathcal{O}_{A\cup\{k\}} = (O_{A\cup\{k\}}; \circ, \zeta, \tau, \Delta, e_1^{2,A\cup\{k\}})$$

I_A^+ is closed w.r.t. ζ, τ and $e_1^{2,A\cup\{k\}}$:

- $e_1^{2,A\cup\{k\}} = (e_1^{2,A})^+$
- $\zeta(f) = (\zeta(f^-))^+$
- $\tau(f) = (\tau(f^-))^+$

I_A^+ is not closed w.r.t. Δ and $\circ$:

- $\Delta(f) \neq (\Delta(f^-))^+$
- $f \circ g \neq ((f^-) \diamond (g^-))^+$

$\Delta(f) \neq (\Delta(f^-))^+$

f	0	1	2
0	0	0	0
1	2	0	2
2	2	0	2

	Δf
0	0
1	0
2	2

f^-	0	1
0	0	0
1	2	0

	Δf^-
0	0
1	0

	$(\Delta f^-)^+$
0	0
1	0
2	0

- $A = \{0,1\} \Rightarrow f \circ g = ((f^-) \diamond (g^-))^+$
- $|A| \geq 3 \Rightarrow f \circ g \neq ((f^-) \diamond (g^-))^+$

f	0	1	2	3
0			1	
1			1	
2			2	
3			3	

	g
0	0
1	0
2	1
3	3

$\Rightarrow$

$f \circ g$	0	1	2	3
0			1	
1			1	
2			1	
3			3	

$(f \circ g)(3,2) = f(g(3),2) = 3$

f^-	0	1	2
0			1
1			1
2			2

	g^-
0	0
1	0
2	1

$\Rightarrow$

$(f^- \diamond g^-)^+$	0	1	2	3
0			1	
1			1	
2			1	
3			1	

$$\begin{aligned}(f^- \diamond g^-)^+(3,2) &= f(g(0),2) \sqcap f(g(1),2) \sqcap f(g(2),2) \\ &= f(0,2) \sqcap f(0,2) \sqcap f(1,2) = 1\end{aligned}$$

Algebra of extended IS operations

Full algebra of extended IS operations:

$$\mathcal{I}_A^+ = (I_A^+; \circ_i, \zeta, \tau, \Delta_i, e_1^{2,A\cup\{k\}})$$

where

$$\Delta_i(f) = (\Delta(f^-))^+$$

$$f \circ_i g = ((f^-) \diamond (g^-))^+$$

Extended IS clone

$$\mathcal{I}_A^+ = (I_A^+; \circ_i, \zeta, \tau, \Delta_i, e_1^2)$$

Theorem

$C \subseteq I_A^+$ is extended from an IS clone
iff
C is a subuniverse of the full algebra $\mathcal{I}_A^+$ of extended IS operations.

Extended IS clone

i-composition of extended IS operations:
$f \in (I_A^+)^{(n)}, g_1, \ldots, g_n \in (I_A^+)^{(m)}$ $f[g_1, \ldots, g_n] \in (I_A^+)^{(m)}$:

$$f[g_1, \ldots, g_n] = (f^-[g_1^-, \ldots, g_n^-])^+$$

Theorem

$C \subseteq I_A^+$ is an extended clone of IS operations if and only if

- *C contains all projections*
- *C is closed with respect to i-composition.*

Extended IS operations preserving relations

$A = \{0, 1\}$
$\rho \subseteq \{0, 1, 2\}^m \qquad f \in (I_A^+)^{(n)}$

$$\delta(f) = \{(\delta_\alpha(f^-))^+ \mid \alpha : \{1, \ldots, n\} \to \{1, \ldots l\}, 1 \leq l \leq n\}$$

where

$$\delta_\alpha(f^-)(x_1, \ldots, x_l) = f^-(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$$

Definition

f i-preserves ρ if and only if $\delta(f) \subseteq Pol\rho$.

$iPol\rho = \{f \in I_A^+ : f$ i-preserves $\rho\}$

Theorem (Tarasov 1974)

Let $A = \{0, 1\}$. If $C^+ = iPol\rho$, then C is an IS clone.

Thank you for your attention!