Implicit definition of the quaternary discriminator

Miguel Campercholi and Diego Vaggione

FaMAF
Universidad Nacional de Córdoba
Argentina

Szeged, June 2012

Let $\mathcal{K}$ be a class of algebras. Suppose the system of equations

$$\begin{gathered} t_1(x_1,\ldots,x_n,z) = s_1(x_1,\ldots,x_n,z) \\ \vdots \\ t_k(x_1,\ldots,x_n,z) = s_k(x_1,\ldots,x_n,z) \end{gathered}$$

is such that

$$\mathcal{K} \models \forall \overline{x} \exists ! z \bigwedge t_i(\overline{x},z) = s_i(\overline{x},z)$$

Let $\mathcal{K}$ be a class of algebras. Suppose the system of equations

$$\begin{gathered} t_1(x_1, \ldots, x_n, z) = s_1(x_1, \ldots, x_n, z) \\ \vdots \\ t_k(x_1, \ldots, x_n, z) = s_k(x_1, \ldots, x_n, z) \end{gathered}$$

is such that

$$\mathcal{K} \models \forall \overline{x} \exists ! z \bigwedge t_i(\overline{x}, z) = s_i(\overline{x}, z)$$

Then for every $\mathbf{A} \in \mathcal{K}$, it *implicitly defines* a function $f : A^n \to A$

$$f(\overline{a}) = \text{unique } b \text{ such that } \bigwedge t_i^{\mathbf{A}}(\overline{a}, b) = s_i^{\mathbf{A}}(\overline{a}, b)$$

Examples

- Let $\mathbf{G} = \langle G, \cdot, e \rangle$ be a group. The system

$$x \cdot z = e$$

implicitly defines the inverse operation on G.

Examples

- Let $\mathbf{G} = \langle G, \cdot, e \rangle$ be a group. The system

  $$x \cdot z = e$$

  implicitly defines the inverse operation on G.
- Let $\mathbf{L} = \langle L, \vee, \wedge, 0, 1 \rangle$ be a boolean lattice. The system

  $$x \vee z = 1$$
  $$x \wedge z = 0$$

  defines the complement operation on L.

These functions have some nice properties. E.g., they:

- preserve homomorphisms,

These functions have some nice properties. E.g., they:

- preserve homomorphisms,
- preserve products.

These functions have some nice properties. E.g., they:

- preserve homomorphisms,
- preserve products.

These functions have some nice properties. E.g., they:

- preserve homomorphisms,
- preserve products.

We studied them in [Cam&Vag2011a], [Cam&Vag2011b] and [Cam&Vag2011c], where we found all such functions for:

- Boolean algebras, distributive lattices, Kleene algebras, Stone algebras, Tarski algebras, semilattices.
- Algebraically closed fields, finitely generated abelian groups.
- Quasiprimal algebras.
- Algebras with the discriminator implicitly definable.

The *quaternary discriminator* on a set A is the function

$$\mathrm{d}^A(x, y, z, w) = \begin{cases} z & \text{if } x = y, \\ w & \text{if } x \neq y. \end{cases}$$

The *quaternary discriminator* on a set A is the function

$$\mathrm{d}^A(x,y,z,w) = \begin{cases} z & \text{if } x = y, \\ w & \text{if } x \neq y. \end{cases}$$

Problem

When is the quaternary discriminator implicitly definable on every algebra in a class $\mathcal{K}$ (by the same system)?

Let $\mathcal{Q}$ be a quasivariety and let $\mathbf{A} \in \mathcal{Q}$.

- $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) := \{\theta \in \mathrm{Con}(\mathbf{A}) \mid \mathbf{A}/\theta \in \mathcal{Q}\}$

Let $\mathcal{Q}$ be a quasivariety and let $\mathbf{A} \in \mathcal{Q}$.

- $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) := \{\theta \in \mathrm{Con}(\mathbf{A}) \mid \mathbf{A}/\theta \in \mathcal{Q}\}$
- $\theta^{\mathbf{A}}_{\mathcal{Q}}(a, b) := \bigcap\{\theta \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) \mid \langle a, b\rangle \in \theta\}$

Let $\mathcal{Q}$ be a quasivariety and let $\mathbf{A} \in \mathcal{Q}$.

- $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) := \{\theta \in \mathrm{Con}(\mathbf{A}) \mid \mathbf{A}/\theta \in \mathcal{Q}\}$
- $\theta^{\mathbf{A}}_{\mathcal{Q}}(a, b) := \bigcap\{\theta \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) \mid \langle a, b\rangle \in \theta\}$
- $\mathbf{A}$ is *Relatively Simple (RS)* if $|\mathrm{Con}_{\mathcal{Q}}(\mathbf{A})| \leq 2$
  $\mathcal{Q}_{RS} := \{\mathbf{A} \in \mathcal{Q} \mid \mathbf{A} \text{ is RS}\}$

Let $\mathcal{Q}$ be a quasivariety and let $\mathbf{A} \in \mathcal{Q}$.

- $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) := \{\theta \in \mathrm{Con}(\mathbf{A}) \mid \mathbf{A}/\theta \in \mathcal{Q}\}$
- $\theta^{\mathbf{A}}_{\mathcal{Q}}(a, b) := \bigcap\{\theta \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) \mid \langle a, b\rangle \in \theta\}$
- $\mathbf{A}$ is *Relatively Simple (RS)* if $|\mathrm{Con}_{\mathcal{Q}}(\mathbf{A})| \leq 2$
  $\mathcal{Q}_{RS} := \{\mathbf{A} \in \mathcal{Q} \mid \mathbf{A} \text{ is RS}\}$
- $\mathcal{Q}$ has *Equationally Definable Relative Principal Congruences (EDRPC)* if there are quaternary terms $p_1, \dots, p_n, q_1, \dots, q_n$ such that $\forall \mathbf{A} \in \mathcal{Q}$

$$\theta^{\mathbf{A}}_{\mathcal{Q}}(a, b) = \{\langle c, d\rangle : \bigwedge p_i\,(a, b, c, d) = q_i\,(a, b, c, d)\}.$$

Main Theorem

Theorem

Let $\mathcal{K}$ be class of algebras. T.f.a.e.:

1. *The quaternary discriminator is implicitly definable in $\mathcal{K}$.*

Main Theorem

Theorem

Let $\mathcal{K}$ be class of algebras. T.f.a.e.:

1. *The quaternary discriminator is implicitly definable in $\mathcal{K}$.*
2. $\mathbb{S}\mathbb{P}_u(\mathcal{K}) \subseteq \mathcal{Q}_{RS}$ *for some rel. congruence dist. quasivariety $\mathcal{Q}$.*

Main Theorem

Theorem

Let $\mathcal{K}$ be class of algebras. T.f.a.e.:

1. *The quaternary discriminator is implicitly definable in $\mathcal{K}$.*
2. $\mathbb{SP}_u(\mathcal{K}) \subseteq \mathcal{Q}_{RS}$ *for some rel. congruence dist. quasivariety $\mathcal{Q}$.*
3. $\mathcal{K} \subseteq \mathcal{Q}_{RS}$ *for some quasivariety $\mathcal{Q}$ with EDRPC.*

Main Theorem

Theorem

Let $\mathcal{K}$ be class of algebras. T.f.a.e.:

1. *The quaternary discriminator is implicitly definable in $\mathcal{K}$.*
2. $\mathbb{SP}_u(\mathcal{K}) \subseteq \mathcal{Q}_{RS}$ *for some rel. congruence dist. quasivariety $\mathcal{Q}$.*
3. $\mathcal{K} \subseteq \mathcal{Q}_{RS}$ *for some quasivariety $\mathcal{Q}$ with EDRPC.*
4. *There are terms $p_1, \ldots, p_n, q_1, \ldots, q_n$, such that*

$$\mathcal{K} \models \left(\bigwedge p_i(x, y, z, w) = q_i(x, y, z, w) \right) \leftrightarrow (x = y \rightarrow z = w).$$

Main Theorem

Theorem

Let $\mathcal{K}$ be class of algebras. T.f.a.e.:

1. *The quaternary discriminator is implicitly definable in $\mathcal{K}$.*
2. $\mathbb{SP}_u(\mathcal{K}) \subseteq \mathcal{Q}_{RS}$ *for some rel. congruence dist. quasivariety $\mathcal{Q}$.*
3. $\mathcal{K} \subseteq \mathcal{Q}_{RS}$ *for some quasivariety $\mathcal{Q}$ with EDRPC.*
4. *There are terms $p_1, \dots, p_n, q_1, \dots, q_n$, such that*

$$\mathcal{K} \models \left(\bigwedge p_i(x, y, z, w) = q_i(x, y, z, w) \right) \leftrightarrow (x = y \rightarrow z = w).$$

5. *For every trivial satisfiable open formula $O(x_1, \dots, x_m)$ there are terms $s_1, \dots, s_k, t_1, \dots, t_k$ such that*

$$\mathcal{K} \models \left(\bigwedge s_i(\overline{x}) = t_i(\overline{x}) \right) \leftrightarrow O(\overline{x}).$$

Proof.

[1⇒2] Let $\varepsilon(x, y, z, w, u)$ be a conjunction of equations such that for all $\mathbf{A} \in \mathcal{K}$

$$\mathbf{A} \models \varepsilon(x, y, z, w, u) \leftrightarrow \mathrm{d}^{A}(x, y, z, w) = u.$$

Proof.

[1⇒2] Let $\varepsilon(x, y, z, w, u)$ be a conjunction of equations such that for all $\mathbf{A} \in \mathcal{K}$

$$\mathbf{A} \models \varepsilon(x, y, z, w, u) \leftrightarrow \mathrm{d}^{A}(x, y, z, w) = u.$$

This also holds for all $\mathbf{A} \in \mathbb{S}\mathbb{P}_u(\mathcal{K})$.

Main Theorem
Sketching an implication

Proof.

[1⇒2] Let $\varepsilon(x, y, z, w, u)$ be a conjunction of equations such that for all $\mathbf{A} \in \mathcal{K}$

$$\mathbf{A} \vDash \varepsilon(x, y, z, w, u) \leftrightarrow \mathrm{d}^{A}(x, y, z, w) = u.$$

This also holds for all $\mathbf{A} \in \mathbb{SP}_u(\mathcal{K})$. So, if $\mathbf{A}, \mathbf{B} \in \mathbb{SP}_u(\mathcal{K})$ and $h : \mathbf{A} \to \mathbf{B}$, then $h : \langle \mathbf{A}, \mathrm{d}^{A} \rangle \to \langle \mathbf{B}, \mathrm{d}^{B} \rangle$.

Proof.

[1⇒2] Let $\varepsilon(x, y, z, w, u)$ be a conjunction of equations such that for all $\mathbf{A} \in \mathcal{K}$

$$\mathbf{A} \vDash \varepsilon(x, y, z, w, u) \leftrightarrow \mathrm{d}^{A}(x, y, z, w) = u.$$

This also holds for all $\mathbf{A} \in \mathbb{SP}_u(\mathcal{K})$. So, if $\mathbf{A}, \mathbf{B} \in \mathbb{SP}_u(\mathcal{K})$ and $h : \mathbf{A} \to \mathbf{B}$, then $h : \langle \mathbf{A}, \mathrm{d}^{A} \rangle \to \langle \mathbf{B}, \mathrm{d}^{B} \rangle$. Thus h is either injective or constant, and hence

$$\mathbb{SP}_u(\mathcal{K}) \subseteq \mathbb{Q}(\mathcal{K})_{RS}.$$

Main Theorem

Sketching an implication

Proof.

[1⇒2] Let $\varepsilon(x, y, z, w, u)$ be a conjunction of equations such that for all $\mathbf{A} \in \mathcal{K}$

$$\mathbf{A} \vDash \varepsilon(x, y, z, w, u) \leftrightarrow \mathrm{d}^{A}(x, y, z, w) = u.$$

This also holds for all $\mathbf{A} \in \mathbb{SP}_u(\mathcal{K})$. So, if $\mathbf{A}, \mathbf{B} \in \mathbb{SP}_u(\mathcal{K})$ and $h : \mathbf{A} \to \mathbf{B}$, then $h : \langle \mathbf{A}, \mathrm{d}^{A} \rangle \to \langle \mathbf{B}, \mathrm{d}^{B} \rangle$. Thus h is either injective or constant, and hence

$$\mathbb{SP}_u(\mathcal{K}) \subseteq \mathbb{Q}(\mathcal{K})_{RS}.$$

Observe that for all $\mathbf{A} \in \mathcal{K}$

$$\mathbf{A} \vDash \varepsilon(x, y, z, w, z) \leftrightarrow (x = y \vee z = w)\,.$$

Proof.

[1⇒2] Let $\varepsilon(x, y, z, w, u)$ be a conjunction of equations such that for all $\mathbf{A} \in \mathcal{K}$

$$\mathbf{A} \vDash \varepsilon(x, y, z, w, u) \leftrightarrow \mathrm{d}^{A}(x, y, z, w) = u.$$

This also holds for all $\mathbf{A} \in \mathbb{SP}_u(\mathcal{K})$. So, if $\mathbf{A}, \mathbf{B} \in \mathbb{SP}_u(\mathcal{K})$ and $h : \mathbf{A} \to \mathbf{B}$, then $h : \langle \mathbf{A}, \mathrm{d}^{A} \rangle \to \langle \mathbf{B}, \mathrm{d}^{B} \rangle$. Thus h is either injective or constant, and hence

$$\mathbb{SP}_u(\mathcal{K}) \subseteq \mathbb{Q}(\mathcal{K})_{RS}.$$

Observe that for all $\mathbf{A} \in \mathcal{K}$

$$\mathbf{A} \vDash \varepsilon(x, y, z, w, z) \leftrightarrow (x = y \vee z = w).$$

So, by [Cze&Dzi1990], $\mathbb{Q}(\mathcal{K})$ is rel. congruence distributive. $\square$

$\mathcal{Q}$ has *Equationally Definable Principal Meets (EDPM)* if there are quaternary terms $p_1, \ldots, p_n, q_1, \ldots, q_n$ such that for all $\mathbf{A} \in \mathcal{Q}$

$$\theta^{\mathbf{A}}_{\mathcal{Q}}(a,b) \cap \theta^{\mathbf{A}}_{\mathcal{Q}}(c,d) = \bigsqcup \theta^{\mathbf{A}}_{\mathcal{Q}}(p_i(a,b,c,d), q_i(a,b,c,d)).$$

$\mathcal{Q}$ has *Equationally Definable Principal Meets (EDPM)* if there are quaternary terms $p_1, \ldots, p_n, q_1, \ldots, q_n$ such that for all $\mathbf{A} \in \mathcal{Q}$

$$\theta^{\mathbf{A}}_{\mathcal{Q}}(a,b) \cap \theta^{\mathbf{A}}_{\mathcal{Q}}(c,d) = \bigsqcup \theta^{\mathbf{A}}_{\mathcal{Q}}(p_i(a,b,c,d), q_i(a,b,c,d)).$$

Corollary

Let $\mathcal{Q}$ be a quasivariety. The following are equivalent:

1. *$\mathcal{Q}$ is relatively semisimple and has EDRPC.*
2. *$\mathcal{Q}$ has EDPM and $\mathcal{Q}_{RFSI} = \mathcal{Q}_{RS}$.*
3. *$\mathcal{Q}$ has EDPM and $\theta^{\mathbf{A}}_{\mathcal{Q}}(a,b)$ is a complemented element of $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A})$, for every $\mathbf{A} \in \mathcal{Q}$, and $a, b \in A$.*
4. *$\mathcal{Q} = Q(\mathcal{K})$, for some class $\mathcal{K}$ satisfying some of the equivalent conditions of the theorem.*

Relative permutability

An algebra $\mathbf{A} \in \mathcal{Q}$ is *relatively permutable* if

$$\theta \circ \delta = \delta \circ \theta = \theta \sqcup \delta, \text{ for all } \theta, \delta \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A}).$$

Relative permutability

An algebra $\mathbf{A} \in \mathcal{Q}$ is *relatively permutable* if

$$\theta \circ \delta = \delta \circ \theta = \theta \sqcup \delta, \text{ for all } \theta, \delta \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A}).$$

Theorem

Let $\mathcal{Q}$ be rel. semisimple with EDRPC. Let $\varepsilon(x, y, z, w, u)$ be a conj. of equations such that for all $\mathbf{S} \in \mathcal{Q}_{RS}$

$$\mathbf{S} \vDash \varepsilon(x, y, z, w, u) \leftrightarrow \mathrm{d}^{A}(x, y, z, w) = u.$$

Then, for $\mathbf{A} \in \mathcal{Q}$, t.f.a.e.:

1. $\mathbf{A}$ *is relatively permutable.*

Relative permutability

An algebra $\mathbf{A} \in \mathcal{Q}$ is *relatively permutable* if

$$\theta \circ \delta = \delta \circ \theta = \theta \sqcup \delta, \text{ for all } \theta, \delta \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A}).$$

Theorem

Let $\mathcal{Q}$ be rel. semisimple with EDRPC. Let $\varepsilon(x, y, z, w, u)$ be a conj. of equations such that for all $\mathbf{S} \in \mathcal{Q}_{RS}$

$$\mathbf{S} \vDash \varepsilon(x, y, z, w, u) \leftrightarrow \mathrm{d}^{A}(x, y, z, w) = u.$$

Then, for $\mathbf{A} \in \mathcal{Q}$, t.f.a.e.:

1. $\mathbf{A}$ *is relatively permutable.*
2. $\mathbf{A} \vDash \forall xyzw \exists! u \; \varepsilon(x, y, z, w, u)$.

Relative permutability

An algebra $\mathbf{A} \in \mathcal{Q}$ is *relatively permutable* if

$$\theta \circ \delta = \delta \circ \theta = \theta \sqcup \delta, \text{ for all } \theta, \delta \in \mathrm{Con}_{\mathcal{Q}}(\mathbf{A}).$$

Theorem

Let $\mathcal{Q}$ be rel. semisimple with EDRPC. Let $\varepsilon(x, y, z, w, u)$ be a conj. of equations such that for all $\mathbf{S} \in \mathcal{Q}_{RS}$

$$\mathbf{S} \models \varepsilon(x, y, z, w, u) \leftrightarrow \mathrm{d}^{A}(x, y, z, w) = u.$$

Then, for $\mathbf{A} \in \mathcal{Q}$, t.f.a.e.:

1. $\mathbf{A}$ *is relatively permutable.*
2. $\mathbf{A} \models \forall xyzw \exists! u \; \varepsilon(x, y, z, w, u)$.
3. $\mathbf{A}$ *is a boolean product with factors in* $\mathcal{Q}_{RS}$.

So, if $\mathbf{A}$ is relatively permutable we can define a 'new' operation by

$$\mathrm{N}^{\mathbf{A}}(a, b, c, d) = \text{the unique } u \in A \text{ such that } \mathbf{A} \vDash \varepsilon(a, b, c, d, u).$$

So, if **A** is relatively permutable we can define a 'new' operation by

$$\mathrm{N}^{\mathbf{A}}(a, b, c, d) = \text{the unique } u \in A \text{ such that } \mathbf{A} \vDash \varepsilon(a, b, c, d, u).$$

Then:

- $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) = \mathrm{Con}\left(\mathbf{A}, \mathrm{N}^{\mathbf{A}}\right)$.

So, if **A** is relatively permutable we can define a 'new' operation by

$$\mathrm{N}^{\mathbf{A}}(a, b, c, d) = \text{the unique } u \in A \text{ such that } \mathbf{A} \vDash \varepsilon(a, b, c, d, u).$$

Then:

- $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) = \mathrm{Con}\left(\mathbf{A}, \mathrm{N}^{\mathbf{A}}\right)$.
- If $\theta \in \mathrm{Con}\left(\mathbf{A}, \mathrm{N}^{\mathbf{A}}\right)$ then $\mathbf{A}/\theta$ is rel. permutable.

So, if $\mathbf{A}$ is relatively permutable we can define a 'new' operation by

$$\mathrm{N}^{\mathbf{A}}(a, b, c, d) = \text{the unique } u \in A \text{ such that } \mathbf{A} \vDash \varepsilon(a, b, c, d, u).$$

Then:

- $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) = \mathrm{Con}\left(\mathbf{A}, \mathrm{N}^{\mathbf{A}}\right)$.
- If $\theta \in \mathrm{Con}\left(\mathbf{A}, \mathrm{N}^{\mathbf{A}}\right)$ then $\mathbf{A}/\theta$ is rel. permutable.
- $\mathcal{P} = \left\{\left(\mathbf{A}, \mathrm{N}^{\mathbf{A}}\right) : \mathbf{A} \in \mathcal{Q} \text{ is rel. perm.}\right\}$ is a variety.

So, if $\mathbf{A}$ is relatively permutable we can define a 'new' operation by

$$\mathrm{N}^{\mathbf{A}}(a, b, c, d) = \text{the unique } u \in A \text{ such that } \mathbf{A} \vDash \varepsilon(a, b, c, d, u).$$

Then:

- $\mathrm{Con}_{\mathcal{Q}}(\mathbf{A}) = \mathrm{Con}\left(\mathbf{A}, \mathrm{N}^{\mathbf{A}}\right)$.
- If $\theta \in \mathrm{Con}\left(\mathbf{A}, \mathrm{N}^{\mathbf{A}}\right)$ then $\mathbf{A}/\theta$ is rel. permutable.
- $\mathcal{P} = \left\{\left(\mathbf{A}, \mathrm{N}^{\mathbf{A}}\right) : \mathbf{A} \in \mathcal{Q} \text{ is rel. perm.}\right\}$ is a variety.
- Every $\mathbf{A} \in \mathcal{Q}$ has a unique permutable extension $\mathbf{E}$ satisfying $\langle A \rangle^{\mathbf{E}} = E$.

References

- Campercholi, M., Vaggione D.: Algebraic functions, Studia Logica, **98**, 285–306 (2011)
- Campercholi, M., Vaggione D.: An implicit function theorem for algebraically closed fields, Algebra Universalis **65**, 299–304 (2011)
- Campercholi, M., Vaggione D.: Axiomatizability by $\forall\exists!$-sentences, Arch. Math. Logic, **50**, 713–725 (2011)
- Czelakowski, J., Dziobiak, W.: Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class, Algebra Universalis **27**, 128–149 (1990)

THANK YOU!