Algebraic reduction of CSP to digraphs (joint work with D. Delić, M. Jackson and T. Niven)

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Charles University in Prague

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Dedicated to the 80th birthday of Béla Csákány

Outline

Some definitions

- a relational structure: $\mathbb{A} = \langle A; R_1, \ldots, R_n \rangle$, where $R_i \subseteq A^{k_i}$
- a digraph: $\mathbb{G} = \langle G; \rightarrow \rangle$, where $\rightarrow$ is binary

"Everything is finite." – L. Barto

- for a fixed $\mathbb{A}$, $\mathrm{CSP}(\mathbb{A}) = \{\mathbb{X} : \mathbb{X} \rightarrow \mathbb{A}\}$
- complexity of the membership problem?

Conjecture (CSP dichotomy conjecture – Feder, Vardi)

For every $\mathbb{A}$, CSP($\mathbb{A}$) is in P or NP-complete.

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Algebra of polymorphisms

Definition

An operation $f : A^n \to A$ is a polymorphism of $\mathbb{A}$, if f preserves every relation of $\mathbb{A}$.

e.g., for digraphs:

$$\begin{array}{ccccccc} f(a_1 & a_2 & \dots & a_n) & = & a \\ \downarrow & \downarrow & & \downarrow & \Longrightarrow & \downarrow \\ f(b_1 & b_2 & \dots & b_n) & = & b \end{array}$$

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Algebraic approach

For core $\mathbb{A}$, complexity of $\texttt{CSP}(\mathbb{A})$ depends (up to $\texttt{L}$ reductions) only on the idempotent Maltsev conditions satisfied by $\mathbb{A}$.

Conjecture (Jeavons, Bulatov, Krokhin'05)

For core $\mathbb{A}$, CSP($\mathbb{A}$) is in P iff $\mathbb{A}$ is Taylor, i.e., satisfies some nontrivial idempotent Maltsev condition.

$$\text{studying } \texttt{CSP}(\mathbb{A}) \;\downarrow\uparrow\; \text{understanding Maltsev conditions in relational structures}$$

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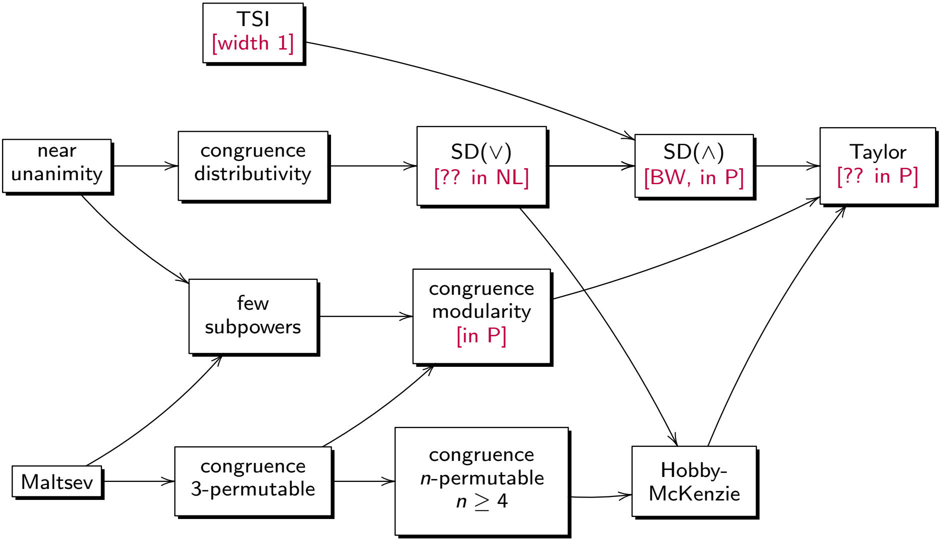

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Reduction to digraphs

Theorem (Feder, Vardi'93)

For every finite relational structure $\mathbb{A}$ there exists a digraph $\mathbb{G}$ [balanced, of height 5] such that $\mathtt{CSP}(\mathbb{A})$ is P-equivalent to $\mathtt{CSP}(\mathbb{G})$.

How much of the algebraic structure is preserved?

- Taylor term (if the CSP dichotomy conjecture holds)
- $SD(\wedge)$ (Bounded width)

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Conjecture (Marković's conjecture)

For digraphs, Maltsev implies majority (3-ary near-unanimity).

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Our result: this is the only new implication between "interesting" Maltsev conditions in digraphs.

Condensation of Maltsev conditions

- Finite algebras

  $\Downarrow$ CD $\Rightarrow$ NU (Barto'11,Zhuk'12),
  CM $\Rightarrow$ Few subpowers (Barto'12)

- Relational structures

  $\Downarrow$ no new implications here

- Binary relations only

  $\Downarrow$ Maltsev $\Rightarrow$ majority (Kazda'09)

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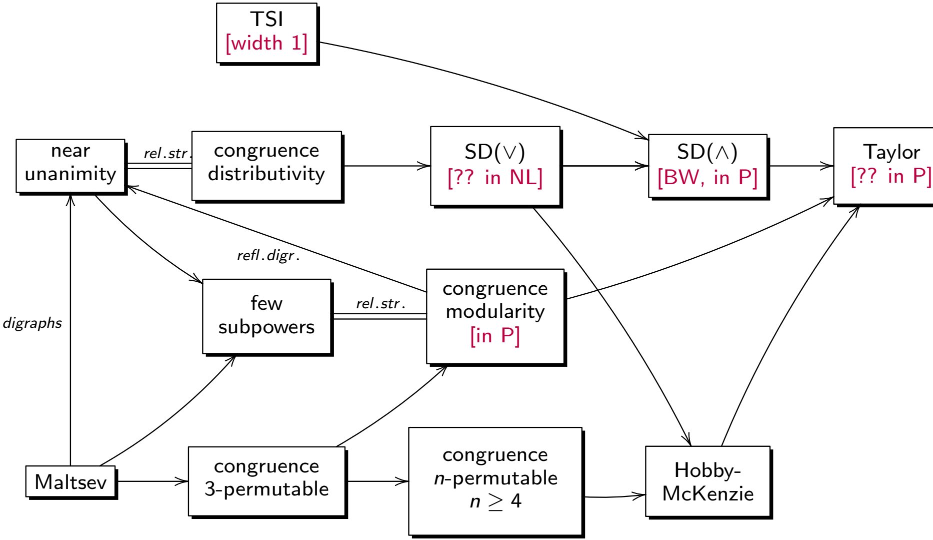
Maltsev conditions: relational structures



The result

Zigzag is the digraph $\begin{tikzcd}[row sep=small, column sep=small] \bullet \arrow[r] & \bullet \\ \bullet \arrow[r] \arrow[ur] & \bullet \end{tikzcd}$. It satisfies all Maltsev conditions from the picture except for the Maltsev Maltsev condition.

Theorem (JB, Delić, Jackson, Niven'11)

For every finite relational structure $\mathbb{A}$ there exists a digraph $\mathbf{D}_{\mathbb{A}}$ such that

1. $\mathrm{CSP}(\mathbb{A})$ *is L-reducible to* $\mathrm{CSP}(\mathbf{D}_{\mathbb{A}})$, $\mathrm{CSP}(\mathbf{D}_{\mathbb{A}})$ *is P-reducible to* $\mathrm{CSP}(\mathbb{A})$
2. $\mathbb{A}$ *is pp-definable from* $\mathbf{D}_{\mathbb{A}}$ *and thus for all Maltsev conditions* Σ
$$\mathbf{D}_{\mathbb{A}} \models \Sigma \;\Rightarrow\; \mathbb{A} \models \Sigma$$
3. *if* Σ *is linear, idempotent, each of its equations is either balanced or in at most 2 variables, and* $\begin{tikzcd}[row sep=small, column sep=small] \bullet \arrow[r] & \bullet \\ \bullet \arrow[r] \arrow[ur] & \bullet \end{tikzcd} \models \Sigma$, *then*
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Remarks

- the construction is quite simple, can be done in $\mathtt{L}$ (wrt. $\mathbb{A}$)
- $\mathbf{D}_{\mathbb{A}}$ is balanced, $\mathbf{D}_{\mathbf{D}_{\mathbb{A}}}$ is balanced of height 5
- among the Maltsev conditions preserved are
  - all Maltsev conditions from the picture except for Maltsev,
  - the six conditions for omitting TCT types,
  - and many more...
  - also, any Maltsev condition satisfied by distributive lattices
- we can construct nice (counter-)examples in digraphs

Problem

Characterize all idempotent Maltsev conditions which are preserved by a reduction (possibly different) from relational structures to digraphs. (Conjecture: all that do not imply Maltsev?)

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The construction

WLOG $\mathbb{A}$ has just one relation R, say n-ary, and no isolated vertices. First, construct the incidence multigraph of $\mathbb{A}$: $\mathbf{Inc}(\mathbb{A}) = \langle A \cup R; E_1, \dots, E_n \rangle$, where $E_i = \{(a, \mathbf{r}) : a = r_i\}$ (the ith projection of R). It's a pp-definition. Now replace multiedges of $\mathbf{Inc}(\mathbb{A})$ with oriented paths. For every $a \in A$ and $\mathbf{r} \in R$ let $\mathbb{P}_{a,\mathbf{r}}$ be the following oriented path:

$$\mathbb{P}_{a,\mathbf{r}} = a \bullet \longrightarrow \bullet \; + \mathbb{P}^1_{a,\mathbf{r}} + \; \bullet \longrightarrow \bullet \; + \cdots + \; \bullet \longrightarrow \bullet \; + \mathbb{P}^n_{a,\mathbf{r}} + \; \bullet \longrightarrow \bullet \, \mathbf{r},$$

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Finally, $\mathbf{D}_{\mathbb{A}}$ is just the union of all the paths $\mathbb{P}_{a,\mathbf{r}}$.

Example: $\mathbb{A} = \langle \{0, 1, 2\}; \{(0, 1, 1), (1, 1, 2)\} \rangle$ [PICTURE].

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$$\mathbb{P}^i_{a,\mathbf{r}} = \begin{cases} \bullet \longrightarrow \bullet & \text{if } (a, \mathbf{r}) \in E_i, \\ \begin{tikzcd}[ampersand replacement=\&] \bullet \arrow[r] \& \bullet \\ \bullet \arrow[r] \arrow[ur] \& \bullet \end{tikzcd} & \text{else.} \end{cases}$$

Finally, $\mathbf{D}_{\mathbb{A}}$ is just the union of all the paths $\mathbb{P}_{a,\mathbf{r}}$.

Example: $\mathbb{A} = \langle \{0,1,2\}; \{(0,1,1),(1,1,2)\} \rangle$ [PICTURE].

The construction

WLOG $\mathbb{A}$ has just one relation R, say n-ary, and no isolated vertices. First, construct the incidence multigraph of $\mathbb{A}$: $\mathbf{Inc}(\mathbb{A}) = \langle A \cup R; E_1, \dots, E_n \rangle$, where $E_i = \{(a, \mathbf{r}) : a = r_i\}$ (the ith projection of R). It's a pp-definition. Now replace multiedges of $\mathbf{Inc}(\mathbb{A})$ with oriented paths. For every $a \in A$ and $\mathbf{r} \in R$ let $\mathbb{P}_{a,\mathbf{r}}$ be the following oriented path:

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Thanks

Thank you for your attention!

Wait!

Bonus: absorption

Let $\mathbb{A}$ be a relational structure and $B \subseteq A$. We say that B is an absorbing subuniverse ($B \trianglelefteq \mathbb{A}$), if B is preserved by all polymorphisms of $\mathbb{A}$, and there exists a polymorphism t such that

$$\begin{aligned} t(A, B, \ldots, B, B) &\subseteq B, \\ t(B, A, \ldots, B, B) &\subseteq B, \\ &\vdots \\ t(B, B, \ldots, A, B) &\subseteq B, \\ t(B, B, \ldots, B, A) &\subseteq B. \end{aligned}$$

Lemma

If $B \trianglelefteq \mathbb{A}$ via k-ary t, then $B \trianglelefteq \mathbf{D}_{\mathbb{A}}$ via some k-ary t'. (Moreover, the construction doesn't add "new" absorption-free subuniverses.)

. . . a few open problems in the absorption theory of Barto and Kozik reduce to digraphs.