Direct products and homomorphisms

Simion Breaz

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Outline

1. Commuting properties
   - Products and coproducts
   - Contravariant functors
   - Covariant functors

Introduction

Important properties of objects in particular categories (e.g. varieties of universal algebras) can be described using commuting properties of some canonical functors. For instance, in [Adámek and Rosicki: Locally presentable categories] there are the following examples:

If $\mathcal{V}$ is a variety of finitary algebras and $A \in \mathcal{V}$ then

- A is *finitely generated* iff the functor $\mathrm{Hom}(A, -) : \mathcal{V} \to \mathbf{Set}$ preserves direct unions (i.e. directed colimits of monomorphisms);
- A is *finitely presented* (i.e. it is generated by finitely many generators modulo finitely many relations) iff the functor $\mathrm{Hom}(A, -) : \mathcal{V} \to \mathbf{Set}$ preserves directed colimits.

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Direct products and Hom-covariant.

Basic constructions are described using **universal properties**. For instance, if $\mathfrak{F} = (A_i)_{i \in I}$ is a family of objects, we can define

- the *direct product* of $\mathfrak{F}$: an object $P := \prod_{i \in I} A_i$ together with a family of homomorphisms $p_i : P \to A_i$ with the (universal) property that for every object B and homomorphisms $\alpha_i : B \to A_i$ there is a unique $\alpha : B \to P$ s.t. $\alpha_i = p_i \alpha$.

Connection with hom-covariant

- P and $p_i : P \to A_i$ represent the direct product of $\mathfrak{F}$ iff for every B the natural map $\mathrm{Hom}(B, P) \to \prod_{i \in I} \mathrm{Hom}(B, A_i)$, $\alpha \mapsto p_i\alpha$ is a bijection, i.e.
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The problem

The general problem

Can we find, for a fixed variety $\mathcal{V}$, a non-trivial object B and (natural) bijective maps $\mathrm{Hom}(\prod A_i, B) \to \prod \mathrm{Hom}(A_i, B)$ for all families (A_i)?

- The answer is NO for sets and vector spaces by computing some cardinalities;

  There are situations when the answer is YES:

Theorem [D.M. Latch, Alg. Univ. (1976)]

In the category of complete ($\vee$-) semilattices arbitrary products and coproducts coincide.

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Contravariant functors acting on direct (co)products

- Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a contravariant functor and $\mathfrak{F} = (A_i)_{i \in I}$ a family of objects in $\mathcal{C}$.
- Assume that
  - there exist the direct products and coproducts for the families $\mathfrak{F}$ and $F(\mathfrak{F}) = (F(A_i))_{i \in I}$.
  - There is a null object 0 such that every composition $A \to 0 \to B$ is the zero homomorphism, hence
  - There are canonical homomorphisms $t_{\mathfrak{F}} : \coprod_{i \in I} A_i \to \prod_{i \in I} A_i$
- Then we have a diagram:

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F(A_i) \arrow[r, "F(f_i')"] \arrow[d, equal] & F(\prod A_i) \arrow[r, "F(\iota_{\mathfrak{F}})"] & F(\coprod A_i) \arrow[r, "F(f_i)"] & F(A_i) \arrow[d, equal] \\
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where the coproducts and products are taken over I, and the maps are the canonical ones.

- The compositions of the maps in top and bottom rows yield the identity maps.
- Using universal properties, this diagram can be completed to a commutative diagram:

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- $\Delta'_{\mathfrak{F}} = \Delta_{\mathfrak{F}} F(\iota_{\mathfrak{F}})$ and $F(\iota_{\mathfrak{F}})\Psi'_{\mathfrak{F}} = \Psi_{\mathfrak{F}}$.
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Contravariant functors acting on direct (co)products

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F(A_i) \arrow[r, "\phi_i"] \arrow[d, equal] & \coprod F(A_i) \arrow[r, equal] \arrow[d, "\Psi'_{\mathfrak{F}}"'] & \coprod F(A_i) \arrow[r, "\gamma_i"] \arrow[d, "\Psi_{\mathfrak{F}}"'] & F(A_i) \arrow[d, equal] \\
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- If $\Psi_{\mathfrak{F}}$ ($\Delta_{\mathfrak{F}}$) are isomorphisms for all $\mathfrak{F}$ we say that F preserves (inverts) direct coproducts.
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The case $F = \mathrm{Hom}_R(-, M)$

- Let R be a unital associative ring. We work on the category of right R-modules.
- $\mathrm{Hom}_R(-, M)$ inverses direct coproducts (the universal property).
- In general, the study of the product inverting property for $\mathrm{Hom}_R(-, M)$ (strongly slender modules) depends on set theoretic axioms:
  - if all cardinals are constructible (V=L) then $\mathbb{Z}$ is strongly slender
  - if there is a non-measurable cardinal then there are not strongly slender abelian groups.
- In general it is enough to work with a weaker notion: a module M is called *slender* if $\mathrm{Hom}_R(-, M)$ inverts direct products with countable many factors.

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- Can we find objects M such that $\mathrm{Hom}_R(-, M)$ preserves products?

Theorem (Goldsmith and Kolmann, J. Alg. '07)

Assume that there is a strongly compact cardinal. If A is an abelian group s.t. $\mathrm{Hom}(\prod A_i, A) \cong \prod \mathrm{Hom}(A_i, A)$ for all families $\mathfrak{F} = (A_i)$ then $A = 0$.

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There is an answer for the natural homomorphisms:

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T.F.A.E. for a right R-module M:

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Covariant functors acting on direct (co)products

- Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant functor and $\mathfrak{F} = (A_i)_{i\in I}$ a family of objects in $\mathcal{C}$.
- Assume that
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F(A_i) \arrow[r, "F(u_i)"] \arrow[d, equal] & F(\coprod A_i) \arrow[r, "F(\iota_{\mathfrak{F}})"] \arrow[d, "\Gamma_{\mathfrak{F}}"'] & F(\prod A_i) \arrow[r, "F(p_i)"] \arrow[d, "\Gamma'_{\mathfrak{F}}"'] & F(A_i) \arrow[d, equal] \\
F(A_i) \arrow[r, "\phi'_i"] & \prod F(A_i) \arrow[r, equal] & \prod F(A_i) \arrow[r, "\gamma'_i"] & F(A_i)
\end{tikzcd}$$
- If $\Phi_{\mathfrak{F}}$ ($\Gamma_{\mathfrak{F}}$) is an isomorphism for all $\mathfrak{F}$ we say that F preserves (inverts) direct coproducts
- If $\Gamma'_{\mathfrak{F}}$ ($\Phi'_{\mathfrak{F}}$) is an isomorphism for all $\mathfrak{F}$ we say that F preserves (inverts) direct products.
- The case $A_i \cong A$ for a fixed $A \mapsto F$, we say that F preserves or inverts self-coproducts/self-products.

logo

The case $F = \mathrm{Hom}_R(M, -)$

- $\mathrm{Hom}_R(M, -)$ preserves products;
- If $\mathrm{Hom}_R(M, -)$ preserves (self-)coproducts then we call M (self-)small;
- Every finitely generated module is small;

Theorem (Rentschler'69; Colpi-Trlifaj'94; Eklof-Goodearl-Trlifaj'97)

- If R is noetherian or perfect then self-small $\Rightarrow$ f.g.
- There are non-f.g. small modules. If R is a simple regular ring s.t. all small modules are f.g. then R is right artinian.

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The case of pointed universal algebras

Let $\mathcal{V}$ be a variety of "pointed" universal algebras: i.e. for every $A \in \mathcal{V}$ we have a fixed singleton subalgebra $0 = \{0\}$, called the point of A, such that the points are invariant under homomorphisms.

- If $\mathfrak{F} = (A_i)_{i\in I}$ is a family of algebras in $\mathcal{V}$, we consider *the restricted direct product*:
  $\prod^{<\omega} A_i = \{(a_i) \in \prod A_i \mid a_i = 0 \text{ for almost all } i \in I\}$.
- There is a canonical map
  $\phi_A^{\mathfrak{F}} : \prod^{<\omega} \mathrm{Hom}(A, A_i) \to \mathrm{Hom}(A, \prod^{<\omega} A_i)$,
  $(f_i) \mapsto [a \mapsto (f_i(a))]$.
- A is called *small* if $\phi_A^{\mathfrak{F}}$ is bijective for all $\mathfrak{F}$.

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If $A \in \mathcal{V}$ and $K \leq A$ we denote by ρ_K the smallest congruence of A which contains $K \times 0$.

- The map $\mathrm{Sub}(A) \to \mathrm{Cong}(A)$, $K \mapsto \rho_K$, is a homomorphism of lattices.
- Conversely, if $\rho \in \mathrm{Cong}(A)$, $\rho \mapsto \rho\langle 0\rangle \in \mathrm{Sub}(A)$.
- $\rho_{\rho\langle 0\rangle} \subseteq \rho$ and $K \subseteq \rho_K\langle 0\rangle$.

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We have the following result:

Theorem

The following are equivalent for a pointed universal algebra $A \in \mathcal{V}$:

- A is small;
- If $(\rho_n)_{n\in\mathbb{N}}$ is a family of congruences such that $\rho_n\langle 0\rangle$ is an increasing chain of subalgebras such that $\cup_{n\in\mathbb{N}}\rho_n\langle 0\rangle = A$ then there is n such that $\rho_n\langle 0\rangle = A$;
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The case of groups

Let G be a group. We consider the category of (left) G-sets, i.e. sets X together with a left action $G \times X \to X$ which is compatible with the monoid structure of G:

- $(gh)x = g(hx)$ and $1x = x$ for all $g, h \in G$ and $x \in X$.
- If we look at G as a G set then the congruences of G are the left congruences of the monoid G. There is a lattice isomorphism $\mathrm{Cong}({}_G G) \cong \mathrm{Sub}(G)$ (G. Bergman).

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- The groups with the property: "If $(H_n)_{n\in\mathbb{N}}$ is an increasing chain of subgroups such that $\cup_{n\in\mathbb{N}} H_n = G$ then there is n such that $H_n = A$" are studied by many authors (cf. G. Bergman, Bull. LMS, 2006).
- These groups are characterized by a commuting property of the functor $X \mapsto X^G = \{x \in X \mid gx = x\}$ with respect direct limits (G. Bergman, J. Alg., 2005).

Open question

- Can we characterize this property by a commuting property of a Hom-functor?

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