# Fully Invariant and Verbal Congruences

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# Verbal Congruences

**A** an algebra, $\Sigma$ a set of equations

$$\lambda_{\Sigma}^{\mathbf{A}} = \mathrm{Cg}^{\mathbf{A}}\Big\{\big(s(a_1,\dots,a_n),\ t(a_1,\dots,a_n)\big) : \\ (s \approx t) \in \Sigma,\ a_1,\dots,a_n \in A\Big\}$$

$\lambda_{\Sigma}^{\mathbf{A}}$ is the smallest congruence $\theta$ such that $\mathbf{A}/\theta \models \Sigma$

$\lambda_{\Sigma}^{\mathbf{A}}$ is the *verbal congruence induced by* $\Sigma$

# Alternate definition

$\mathcal{V}$ a variety (same similarity type as $\mathbf{A}$)
$\Lambda_{\mathcal{V}}^{\mathbf{A}} = \{\, \theta \in \mathrm{Con}(\mathbf{A}) : \mathbf{A}/\theta \in \mathcal{V} \,\}$
$\lambda_{\mathcal{V}}^{\mathbf{A}} = \bigcap \Lambda_{\mathcal{V}}^{\mathbf{A}}$

Easy to see:
- $\mathbf{A}/\lambda_{\mathcal{V}} \in \mathcal{V}$
- If $\mathcal{V} = \mathrm{Mod}(\Sigma)$ then $\lambda_{\mathcal{V}} = \lambda_{\Sigma}$

**Theorem**

*$\theta$ is verbal on $\mathbf{A}$ iff*

$$\mathbf{A}/\psi \in \mathrm{Var}(\mathbf{A}/\theta) \implies \psi \geq \theta.$$

# Examples from Group Theory

Suppose $\Sigma = \{xy \approx yx\}$

On any group **A**, $\lambda_\Sigma^{\mathbf{A}} = \mathrm{Cg}\,\{\,(ab, ba) : a, b \in A\,\}$
corresponds to $\mathbf{A}' = \mathrm{Nml}\,\{\,[a, b] : a, b \in A\,\} = [A, A]$

$\mathbf{A}/\mathbf{A}'$ is the largest Abelian homomorphic image of **A**.

$\Theta_n = \{x^n \approx e\}$

$\lambda^{\mathbf{A}}_{\Theta_n}$ corresponds to $\mathrm{Nml}\{\, a^n : a \in A \,\}$

$\mathbf{A}/\lambda_{\Theta_n}$ is the largest homomorphic image of $\mathbf{A}$ of exponent $n$.

Note: $\lambda_\Sigma \leq \lambda_{\Theta_2}$ since every group of exponent 2 is Abelian

# Fully Invariant Congruences

$\mathrm{End}(\mathbf{A})$ = endomorphism monoid of $\mathbf{A}$

A congruence $\theta$ is *fully invariant* if
$\forall f \in \mathrm{End}(\mathbf{A}) \quad (a, b) \in \theta \implies \big(f(a),\, f(b)\big) \in \theta$

**Theorem**

*Every verbal congruence is fully invariant*

Converse is false

Example: Let $p$ be prime,
$\mathbf{A}$ the Abelian group $\langle a, b \mid pa = p^2 b = 0 \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$.

$$S = \{ x \in A : px = 0 \} \supsetneq T = \{ px : x \in A \}$$

$S$ is fully invariant but

$\mathbf{A}/S \in \mathrm{Var}(\mathbf{A}/T)$ so $S$ not verbal

# General Question

Find conditions under which

$$\text{fully invariant} \implies \text{verbal}$$

- for a congruence
- for an algebra
- for a variety.

An algebra is called *verbose* if every fully invariant congruence is verbal.
A variety is *verbose* if every member is verbose.

# The Monolith

Observation: The monolith of a finite s.i. is fully invariant

Is the monolith of a finite s.i. algebra verbal?

Let **A** be a finite s.i. with monolith $\mu$.
$\mu$ is verbal iff $\mathbf{A} \notin \mathrm{Var}(\mathbf{A}/\mu)$

### Theorem (Kovács and Newman, 1966)

*The monolith of a finite s.i. group is verbal*

### Theorem (Kiss, 1992)

*Let $\mathcal{S}$ be a finite set of finite s.i. algebras such that $HS(\mathcal{S})_{\mathrm{si}} \subseteq S$. If $\mathcal{V} = \mathrm{Var}(\mathcal{S})$ is congruence-modular then for every $\mathbf{A} \in \mathcal{V}_{\mathrm{si}}$, $\mathbf{A}/\mu_{\mathbf{A}} \in \mathrm{Var}\{\, \mathbf{D}/\mu_{\mathbf{D}} : \mathbf{D} \in \mathcal{S} \,\}$.*

**Theorem**

*In a congruence-modular variety, the monolith of a finite s.i. algebra is verbal.*

**This fails for infinite algebras**

Every congruence on the group $\mathbb{Z}(p^\infty)$ is fully invariant.

$$\mathrm{Sub}(\mathbb{Z}(p^\infty)) = \left\langle 0 \right\rangle \subset \left\langle \frac{1}{p} \right\rangle \subset \left\langle \frac{1}{p^2} \right\rangle \subset \cdots \subset \left\langle 1 \right\rangle$$

Only the first and the last are verbal since $\mathbb{Z}(p^\infty)/\langle \frac{1}{p^k} \rangle \cong \mathbb{Z}(p^\infty)$

## Example (Bergman-McKenzie)

There is a 3-element algebra **A**, generating an equationally complete variety, $\mathcal{V}$.
$\mathcal{V}$ contains a 5-element algebra, **B**, that is s.i. but not simple. Thus

$$\mathbf{B} \in \mathcal{V} = \mathrm{Var}(\mathbf{B}/\mu_{\mathbf{B}})$$

So $\mu_{\mathbf{B}}$ is not verbal.

**Theorem**

*Every free algebra is verbose.*

*In fact*

*Every projective algebra is verbose.*

# Verbose Varieties

**Theorem**

*A variety of Abelian groups is verbose if and only if it is of square-free exponent.*

## Lemma

*Let $r\colon \mathbf{A} \to \mathbf{B}$ be a retraction,*
*$s\colon \mathbf{A} \to \mathbf{B}$ any homomorphism,*
*$\theta$ fully invariant on $\mathbf{A}$.*
*Then $\theta \subseteq \ker(r) \implies \theta \subseteq \ker(s)$.*

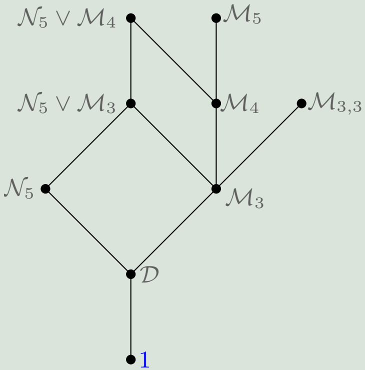
$$\begin{tikzcd}
& \mathbf{B} \arrow[dl, bend right, "j"'] \\
\mathbf{A} \arrow[ur, "r"'] \arrow[dr, "s"'] & \\
& \mathbf{B} \arrow[uu, "id"']
\end{tikzcd}$$

## Theorem

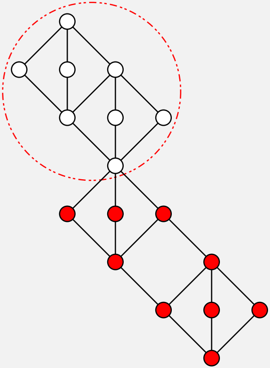
*Let $\mathcal{V}$ be finitely generated, congruence-distributive and $\mathcal{V}_{\mathrm{si}} \subseteq \mathcal{V}_{proj}$. Then $\mathcal{V}$ is verbose.*

## Example

Which small varieties of lattices are verbose?



# Var($\mathbf{M}_{3,3}$), is not verbose



Fully invariant congruence that is not verbose

### Theorem

*Suppose $\mathcal{V}$ contains exactly one s.i. algebra, **P**, which is finite. Then $\mathcal{V}$ is verbose.*

### Corollary

*If $\mathcal{V}$ is finitely generated, congruence-modular and minimal, then $\mathcal{V}$ is verbose.*

**Theorem**

*Let* $\mathbf{A}$ *be a finite simple algebra generating a congruence-modular, Abelian variety,* $\mathcal{V}$*. Then* $\mathcal{V}$ *is verbose.*

**Theorem**

*Every 2-element algebra generates a verbose variety except* $\langle 2, r \rangle$, $\langle 2, r, 0 \rangle$, $\langle 2, r, 1 \rangle$, *and* $\langle 2, r, 0, 1 \rangle$, *where* $r(x) = 1 - x$.

## Theorem

*Let $\mathcal{V}$ be a finitely generated discriminator variety. Then $\mathcal{V}_{fin}$ is verbose.*

## Proof.

$\mathbf{A} \in \mathcal{V}_{\text{fin}} \implies \mathbf{A} \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ all simple.

$\eta_i = \ker(\mathbf{A} \twoheadrightarrow \mathbf{A}_i)$.

Suppose $\theta \in \text{Con}(\mathbf{A})$ not verbal. Then

$\exists i, j \;\; \theta \leq \eta_i, \; \theta \not\leq \eta_j, \; \mathbf{A}_j \overset{h}{\rightarrowtail} \mathbf{A}_i$

Define $e(\mathbf{x}) = (x_1, \ldots, x_{i-1}, h(x_j), x_{i+1}, \ldots, x_n)$.

Then $e \in \text{End}(\mathbf{A})$ but

$(\mathbf{a}, \mathbf{b}) \in \theta - \eta_j \implies \big(e(\mathbf{a}), e(\mathbf{b})\big) \notin \theta$. □

Can this argument be extended to infinite algebras?

Need a representation that is “almost as good” as direct product

Answer: NU-duality

Assume $\mathcal{V} = \mathrm{Var}(\mathbf{M})$, $\mathbf{M}$ subalgebra-primal

$\mathbf{A} \in \mathcal{V} \implies \mathbf{A} \rightsquigarrow \langle X, T, S\rangle$, $T$ a Boolean Topology on $X$.

Instead of constructing endomorphism $\mathbf{A} \xrightarrow{e} \mathbf{A}$
build continuous map $X \xleftarrow{\widehat{e}} X$
Thus $\mathcal{V}$ is verbose

Should be true for finitely generated discriminator variety.

In fact should be true if $\mathrm{Var}(\mathbf{M}) = \mathrm{QVar}(\mathbf{M})$ is semisimple arithmetical.