

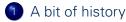
#### CHARACTERISING CATEGORICAL EQUIVALENCE OF FINITE SEMIGROUPS

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Szeged, Hungary, 25 June 2012



# Outline



- 2 Definitions and simple results
- 3 How to prove this?



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- 3 How to prove this?



#### Historical context

1964 John Isbell: Which Lawvere theories have categorically equivalent categories of models?

(in Subobjects, adequacy, completeness, and categories of algebras)

1996 Ralph McKenzie: An algebraic version of categorical equivalence for varieties and more general algebraic theories

$$\mathcal{V} \equiv_{\mathrm{cat}} \mathcal{W} \iff \exists \mathbf{A}, \mathbf{B} \exists e \exists n \in \mathbb{N} \colon \mathbf{B} \cong_{\mathrm{w}} e\left(\mathbf{A}^{[n]}\right) \land$$

 $\mathcal{V} = Var(\mathbf{A}), \mathcal{W} = Var(\mathbf{B})$ 

1997 László Zádori: *Relational sets and categorical equivalence of algebras* (finite algebras)

 $A \equiv_{cat} B \iff \exists \underline{A}, \underline{B}$ : same minimal resets up to isomorphism 1997 László Zádori: *Categorical equivalence of finite groups* 

$$A \equiv_{cat} B \iff A \cong_{w} B$$

2001 Keith Kearnes and Ágnes Szendrei: ideas for Relational Structure Theory (RST)



- 2007/8 Reinhard Pöschel presents fragments of this theory in a lecture on clone theory
- 2008/9 MB ~> Szeged to study this theory (meet Waldhauser & Zádori)
  - 2009 MB: diploma thesis on details of RST
  - 2009 AAA 78, Berne: talk on RST, example of groups, question about monoids
  - 2009 Tamás Waldhauser: solves the question for semigroups
  - 2012 AAA 83, Novi Sad: Oleg Košik, semilattices

 $\textbf{A} \equiv_{\mathrm{cat}} \textbf{B} \iff \textbf{A} \cong_{\mathrm{w}} \textbf{B}$ 

2012 Szeged: finite semigroups

$$\mathbf{A} \equiv_{\operatorname{cat}} \mathbf{B} \iff \mathbf{A} \cong_{\operatorname{w}} \mathbf{B}$$



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#### Categorical equivalence

Definition (Categorical equivalence of categories C and D)  $C \equiv_{cat} D : \iff \exists C \xrightarrow{F} D \exists D \xrightarrow{G} C : F \circ G \cong id_{D} \land G \circ F \cong id_{C}$ 



#### Categorical equivalence

Definition (Categorical equivalence of categories  ${\mathcal C}$  and  ${\mathcal D})$ 

 $\mathcal{C} \equiv_{\mathrm{cat}} \mathcal{D} \quad :\iff \quad \exists \mathcal{C} \xrightarrow{F} \mathcal{D} \exists \mathcal{D} \xrightarrow{G} \mathcal{C} \colon F \circ G \cong \mathrm{id}_{\mathcal{D}} \land G \circ F \cong \mathrm{id}_{\mathcal{C}}$ 

#### Remark

$$\mathcal{C} \equiv_{\operatorname{cat}} \mathcal{D} \iff \operatorname{Skeleton}(\mathcal{C}) \cong \operatorname{Skeleton}(\mathcal{D})$$



#### Categorical equivalence

Definition (Categorical equivalence of categories  $\mathcal C$  and  $\mathcal D)$ 

 $\mathcal{C} \equiv_{\mathrm{cat}} \mathcal{D} \quad : \Longleftrightarrow \quad \exists \mathcal{C} \xrightarrow{F} \mathcal{D} \exists \mathcal{D} \xrightarrow{G} \mathcal{C} \colon F \circ G \cong \mathrm{id}_{\mathcal{D}} \land G \circ F \cong \mathrm{id}_{\mathcal{C}}$ 

#### Remark

$$\mathcal{C} \equiv_{\operatorname{cat}} \mathcal{D} \iff \operatorname{Skeleton}(\mathcal{C}) \cong \operatorname{Skeleton}(\mathcal{D})$$

Definition (Categorical equivalence of algebras **A** and **B**)  $\mathbf{A} \equiv_{\operatorname{cat}} \mathbf{B} : \iff \operatorname{Var}(\mathbf{A}) \equiv_{\operatorname{cat}} \operatorname{Var}(\mathbf{B}) \land F(\mathbf{A}) = \mathbf{B}$ 



#### Weak isomorphism

#### Definition (Term equivalence of algebras **A** and **B**)

$$\mathbf{A} \equiv_{\text{term}} \mathbf{B} : \iff A = B \land \text{Term}(\mathbf{A}) = \text{Term}(\mathbf{B})$$



#### Weak isomorphism

Definition (Term equivalence of algebras **A** and **B**)

 $\mathbf{A} \equiv_{\text{term}} \mathbf{B} : \iff A = B \land \text{Term}(\mathbf{A}) = \text{Term}(\mathbf{B})$ 

Definition (Weak isomorphism between algebras  ${\bf A}$  and  ${\bf B})$ 

$$\mathbf{A} \cong_{\mathrm{w}} \mathbf{B} : \iff \exists \mathbf{B}' \exists \mathbf{A} \xrightarrow{\varphi} \mathbf{B}' \text{ iso: } \mathbf{B}' \equiv_{\mathrm{term}} \mathbf{B}.$$



## Weak isomorphism

Definition (Term equivalence of algebras **A** and **B**)

 $\mathbf{A} \equiv_{\text{term}} \mathbf{B} : \iff A = B \land \text{Term}(\mathbf{A}) = \text{Term}(\mathbf{B})$ 

Definition (Weak isomorphism between algebras A and B)

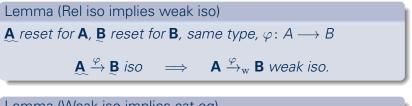
$$\mathbf{A} \cong_{\mathrm{w}} \mathbf{B} : \iff \exists \mathbf{B}' \exists \mathbf{A} \xrightarrow{\varphi} \mathbf{B}' \text{ iso: } \mathbf{B}' \equiv_{\mathrm{term}} \mathbf{B}.$$

Definition (Reset for an algebra **A**)  
Relational structure 
$$\underline{A}$$
 reset for **A**  
 $:\iff$  Term  $(\underline{A}) = \bigcup_{n \in \mathbb{N}}$  Hom  $(\underline{A}^n, \underline{A}) = \text{Pol }\underline{A}$ 

#### **Basic facts**

Lemma (Rel iso implies weak iso)  $\mathbf{A}$  reset for  $\mathbf{A}$ ,  $\mathbf{B}$  reset for  $\mathbf{B}$ , same type,  $\varphi \colon A \longrightarrow B$  $\mathbf{A} \xrightarrow{\varphi} \mathbf{B}$  iso  $\implies \mathbf{A} \xrightarrow{\varphi}_{\mathbf{w}} \mathbf{B}$  weak iso.

#### Basic facts





Converse implication?

#### Basic facts

Lemma (Rel iso implies weak iso)

**A** reset for **A**, **B** reset for **B**, same type,  $\varphi \colon A \longrightarrow B$ 

 $\underbrace{\mathbf{A}}_{\infty} \xrightarrow{\varphi} \underbrace{\mathbf{B}}_{\infty} iso \implies \mathbf{A} \xrightarrow{\varphi}_{w} \mathbf{B} weak iso.$ 



Converse implication? Holds e.g. for

- finite groups (Zádori 1997)
- semilattices? (Košik, AAA 83)
- finite semigroups



# Outline

#### A bit of history

- Definitions and simple results
- 3 How to prove this?



## First step: read Zádori's paper

Theorem (Thm. 2.3 in Zádori, cat. eq. of finite groups, 1997) **A**, **B** two finite algebras.  $\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B} \iff \exists \mathbf{A} \text{ for } \mathbf{A} \exists \mathbf{B} \text{ for } \mathbf{B} \colon \mathrm{type}(\mathbf{A}) = \mathrm{type}(\mathbf{B}) \land$   $``\mathrm{MinRelSets}(\mathbf{A})/\cong'' = ``\mathrm{MinRelSets}(\mathbf{B})/\cong''$ 



## First step: read Zádori's paper

Theorem (Thm. 2.3 in Zádori, cat. eq. of finite groups, 1997)

A, B two finite algebras.

$$\begin{split} \mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B} & \longleftrightarrow \quad \exists \underbrace{\mathbf{A}}_{\sim} \textit{ for } \mathbf{A} \exists \underbrace{\mathbf{B}}_{\sim} \textit{ for } \mathbf{B} : \ \mathsf{type}(\underbrace{\mathbf{A}}) = \mathsf{type}(\underbrace{\mathbf{B}}) \land \\ & \text{``MinRelSets}(\underbrace{\mathbf{A}}) / \cong \text{``} = \text{``MinRelSets}(\underbrace{\mathbf{B}}) / \cong \text{``} \end{split}$$

Minimal resets correspond to certain irreducible neighbourhoods ( = images of idempotent unary terms of  $\mathbf{A}$  / idempotent retracts of  $\mathbf{A}$ ).

Definition (Two sorts of elements in a finite semigroup A)

- $x \in A$  group element : $\iff \exists n \in \mathbb{N} : \langle x \rangle_{\mathbf{A}} \cong \mathbb{Z}_n$  group
- Gr(A) set of all group elements.
- A completely regular : $\iff$  Gr(A) = A.
- For  $x \in A$ , let  $\operatorname{ord}(x) := |\langle x \rangle_{\mathbf{A}}|$ .
- $\exp(\operatorname{Gr}(\mathbf{A})) := \operatorname{lcm} \{ \operatorname{ord}(x) \mid x \in \operatorname{Gr}(\mathbf{A}) \}$

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Proposition (Neighbourhoods of a finite semigroup  $\boldsymbol{\mathsf{A}})$ 

 $\exp(\operatorname{Gr}(\mathbf{A})) = \prod_{i \in I} q_i$  with distinct prime powers  $q_i$ .

- A completely regular  $\implies$  (Neigh  $\mathbf{A}, \subseteq$ )  $\cong$  ( $\mathcal{P}(\{q_i \mid i \in l\}), \subseteq$ )
- A not completely regular

 $\implies (\mathsf{Neigh}\,\mathbf{A},\subseteq) \cong (\mathcal{P}\left(\{\,q_i \mid i \in I\}\right),\subseteq) + \top$ 

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• minimal: atoms,  $\perp$  ( $\top$  if **A** not completely regular)

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- minimal: all pairwise nonisomorphic



### Third step: read Zádori's paper again...

... and generalise to completely regular semigroups

#### Proposition

 $2 \leq |\mathbf{A}| < \infty$  completely regular semigroup. The product of its minimal resets  $\mathbf{U}_1, \ldots, \mathbf{U}_n$  has an  $\subseteq$ -minimal idempotent retract  $\mathbf{U}$  (neighbourhood) w.r.t. containing a certain set S.

$$S \subseteq \mathbf{U} \subseteq \prod_{i=1}^{n} \mathbf{U}_{i} \twoheadrightarrow \mathbf{U}_{i}$$



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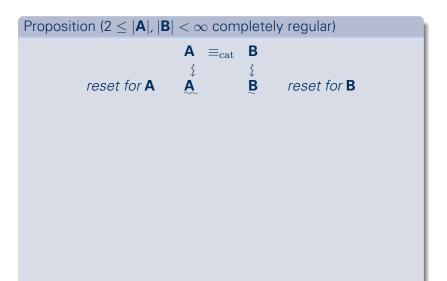
#### Proposition

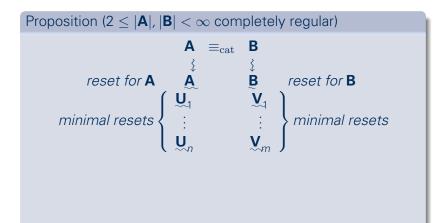
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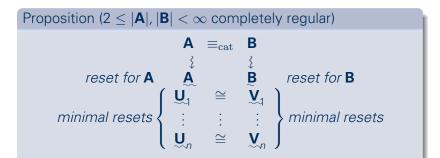
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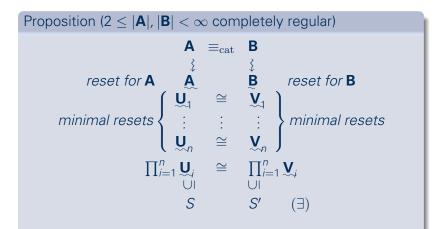
Proposition (2  $\leq$  |**A**|, |**B**|  $< \infty$  completely regular)

 $\textbf{A} ~\equiv_{\rm cat} ~\textbf{B}$ 



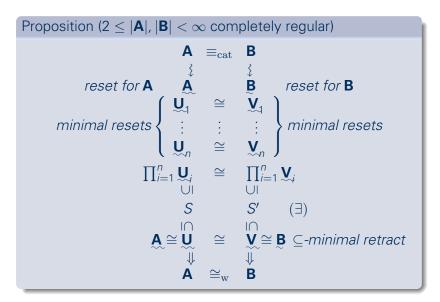






Proposition (2  $\leq$  |**A**|, |**B**|  $< \infty$  completely regular) Α  $\equiv_{cat}$ В  $\begin{array}{cccc} \text{reset for } \mathbf{A} & \overset{\flat}{\mathbf{A}} & \overset{\flat}{\mathbf{B}} & \overset{\flat}{\mathbf{B}} \\ \text{minimal resets} & \overset{\flat}{\mathbf{U}_{1}} & \cong & \overset{\flat}{\mathbf{V}_{1}} \\ \vdots & \vdots & \vdots \\ \overset{\flat}{\mathbf{U}_{n}} & \cong & \overset{\flat}{\mathbf{V}_{n}} \end{array} \end{array}$ reset for **B** minimal resets  $\prod_{i=1}^{n} \bigcup_{i=1}^{n} \bigcup_{i$ S'  $(\exists)$ S  $\mathbf{A} \cong \mathbf{U} \qquad \qquad \mathbf{V} \cong \mathbf{B} \subseteq -minimal \ retract$ 

Proposition (2  $\leq$  |**A**|, |**B**|  $< \infty$  completely regular)  $\mathbf{A} \equiv_{\text{cat}} \mathbf{B}$ reset for  $\mathbf{A}$   $\mathbf{A}$   $\mathbf{B}$  reset for  $\mathbf{B}$ minimal resets  $\left\{ \begin{array}{ccc} \mathbf{V} & \mathbf{S} \\ \mathbf{V}_{1} & \cong & \mathbf{V}_{1} \\ \vdots & \vdots & \vdots \\ \mathbf{U}_{n} & \cong & \mathbf{V}_{n} \end{array} \right\}$  minimal resets  $\prod_{i=1}^{n} \bigcup_{\substack{i=1\\ \cup i}}^{n} \cong \prod_{\substack{i=1\\ \cup i}}^{n} \bigvee_{i}$  $S \qquad S' \quad (\exists)$  $\mathbf{A} \cong \overset{\mathsf{I}}{\mathbf{U}} \cong \overset{\mathsf{I}}{\mathbf{V}} \cong \mathbf{B} \subseteq \text{-minimal retract}$ 





... do something yourself

Proposition ( $\mathbf{A} \equiv_{cat} \mathbf{B}$  both non-completely regular)

$$\textbf{A} ~\equiv_{\mathrm{cat}} ~\textbf{B}$$

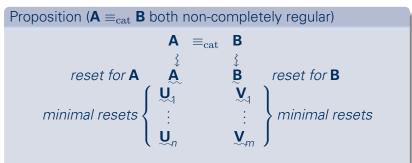


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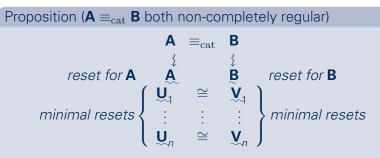
Proposition ( $\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B}$  both non-completely regular)

$$\begin{array}{ccc} A \equiv_{\mathrm{cat}} & B \\ & \downarrow & \downarrow \\ reset \ for \ A & A & B \\ & & & B \\ & & & & B \\ \end{array} \quad reset \ for \ B \end{array}$$

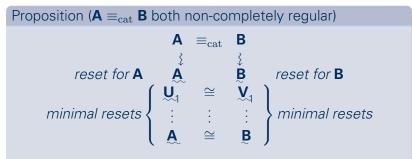




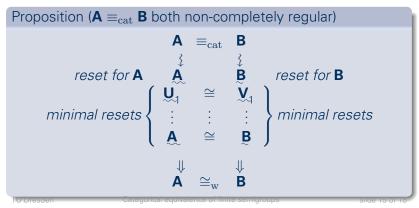






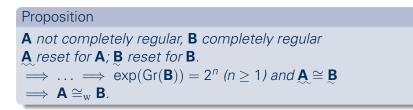








#### Sixth step: solve the mixed case





# Done! Theorem For finite semigroups A and B we have $A \equiv_{cat} B \iff A \cong_{w} B.$



### Thank you for your attention.