TECHNISCHE UNIVERSITÄT DRESDEN

CHARACTERISING CATEGORICAL EQUIVALENCE OF FINITE SEMIGROUPS

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Outline

1. A bit of history
2. Definitions and simple results
3. How to prove this?

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Historical context

1964 John Isbell: Which Lawvere theories have categorically equivalent categories of models?

(in *Subobjects, adequacy, completeness, and categories of algebras*)

1996 Ralph McKenzie: *An algebraic version of categorical equivalence for varieties and more general algebraic theories*

$$\mathcal{V} \equiv_{\mathrm{cat}} \mathcal{W} \iff \exists \mathbf{A}, \mathbf{B} \exists e \exists n \in \mathbb{N} : \mathbf{B} \cong_{\mathrm{w}} e\left(\mathbf{A}^{[n]}\right) \wedge \mathcal{V} = \mathrm{Var}(\mathbf{A}), \mathcal{W} = \mathrm{Var}(\mathbf{B})$$

1997 László Zádori: *Relational sets and categorical equivalence of algebras* (finite algebras)

$$\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B} \iff \exists \underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{B}} : \text{ same minimal resets up to isomorphism}$$

1997 László Zádori: *Categorical equivalence of finite groups*

$$\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B} \iff \mathbf{A} \cong_{\mathrm{w}} \mathbf{B}$$

2001 Keith Kearnes and Ágnes Szendrei: ideas for Relational Structure Theory (RST)

2007/8	Reinhard Pöschel presents fragments of this theory in a lecture on clone theory
2008/9	MB $\rightsquigarrow$ Szeged to study this theory (meet Waldhauser & Zádori)
2009	MB: diploma thesis on details of RST
2009	AAA 78, Berne: talk on RST, example of groups, question about monoids
2009	Tamás Waldhauser: solves the question for semigroups
2012	AAA 83, Novi Sad: Oleg Košik, semilattices

$$\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B} \iff \mathbf{A} \cong_{\mathrm{w}} \mathbf{B}$$

2012 **Szeged: finite semigroups**

$$\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B} \iff \mathbf{A} \cong_{\mathrm{w}} \mathbf{B}$$

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Categorical equivalence

Definition (Categorical equivalence of categories $\mathcal{C}$ and $\mathcal{D}$)

$$\mathcal{C} \equiv_{\mathrm{cat}} \mathcal{D} \quad :\Longleftrightarrow \quad \exists \mathcal{C} \xrightarrow{F} \mathcal{D}\, \exists \mathcal{D} \xrightarrow{G} \mathcal{C} \colon F \circ G \cong \mathrm{id}_{\mathcal{D}} \wedge G \circ F \cong \mathrm{id}_{\mathcal{C}}$$

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Remark

$$\mathcal{C} \equiv_{\mathrm{cat}} \mathcal{D} \quad \Longleftrightarrow \quad \mathrm{Skeleton}\,(C) \cong \mathrm{Skeleton}\,(D)$$

Categorical equivalence

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Remark

$$\mathcal{C} \equiv_{\mathrm{cat}} \mathcal{D} \quad \Longleftrightarrow \quad \mathrm{Skeleton}\,(C) \cong \mathrm{Skeleton}\,(D)$$

Definition (Categorical equivalence of algebras $\mathbf{A}$ and $\mathbf{B}$)

$$\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B} \quad :\Longleftrightarrow \quad \mathrm{Var}(\mathbf{A}) \equiv_{\mathrm{cat}} \mathrm{Var}(\mathbf{B}) \ \wedge \ F(\mathbf{A}) = \mathbf{B}$$

Weak isomorphism

Definition (Term equivalence of algebras $\mathbf{A}$ and $\mathbf{B}$)

$$\mathbf{A} \equiv_{\text{term}} \mathbf{B} \quad :\Longleftrightarrow \quad A = B \wedge \mathrm{Term}(\mathbf{A}) = \mathrm{Term}(\mathbf{B})$$

Weak isomorphism

Definition (Term equivalence of algebras A and B)

$$\mathbf{A} \equiv_{\mathrm{term}} \mathbf{B} \quad :\Longleftrightarrow \quad A = B \wedge \mathrm{Term}\,(\mathbf{A}) = \mathrm{Term}\,(\mathbf{B})$$

Definition (Weak isomorphism between algebras A and B)

$$\mathbf{A} \cong_{\mathrm{w}} \mathbf{B} \quad :\Longleftrightarrow \quad \exists \mathbf{B}'\, \exists \mathbf{A} \xrightarrow{\varphi} \mathbf{B}' \text{ iso} : \quad \mathbf{B}' \equiv_{\mathrm{term}} \mathbf{B}.$$

Weak isomorphism

Definition (Term equivalence of algebras $\mathbf{A}$ and $\mathbf{B}$)

$$\mathbf{A} \equiv_{\mathrm{term}} \mathbf{B} \quad :\Longleftrightarrow \quad A = B \wedge \operatorname{Term}(\mathbf{A}) = \operatorname{Term}(\mathbf{B})$$

Definition (Weak isomorphism between algebras $\mathbf{A}$ and $\mathbf{B}$)

$$\mathbf{A} \cong_{\mathrm{w}} \mathbf{B} \quad :\Longleftrightarrow \quad \exists \mathbf{B}' \, \exists \mathbf{A} \xrightarrow{\varphi} \mathbf{B}' \text{ iso}: \quad \mathbf{B}' \equiv_{\mathrm{term}} \mathbf{B}.$$

Definition (Reset for an algebra $\mathbf{A}$)

Relational structure $\underset{\sim}{\mathbf{A}}$ reset for $\mathbf{A}$
$:\Longleftrightarrow$ $\operatorname{Term}(\mathbf{A}) = \bigcup_{n\in\mathbb{N}} \operatorname{Hom}(\underset{\sim}{\mathbf{A}}^n, \underset{\sim}{\mathbf{A}}) = \operatorname{Pol} \underset{\sim}{\mathbf{A}}$

Basic facts

Lemma (Rel iso implies weak iso)

$\underset{\sim\sim}{\mathbf{A}}$ *reset for* $\mathbf{A}$, $\underset{\sim}{\mathbf{B}}$ *reset for* $\mathbf{B}$, *same type*, $\varphi\colon A \longrightarrow B$

$$\underset{\sim\sim}{\mathbf{A}} \xrightarrow{\varphi} \underset{\sim}{\mathbf{B}} \text{ iso} \quad \Longrightarrow \quad \mathbf{A} \xrightarrow{\varphi}_{\mathrm{w}} \mathbf{B} \text{ weak iso.}$$

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Lemma (Weak iso implies cat eq)

$$\mathbf{A} \cong_{\mathrm{w}} \mathbf{B} \quad \overset{\textit{McKenzie '96}}{\Longrightarrow} \quad \mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B}$$

Converse implication?

Basic facts

Lemma (Rel iso implies weak iso)

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Lemma (Weak iso implies cat eq)

$$\mathbf{A} \cong_{\mathrm{w}} \mathbf{B} \quad \xRightarrow{\text{McKenzie '96}} \quad \mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B}$$

Converse implication? Holds e.g. for

- finite groups (Zádori 1997)
- semilattices? (Košik, AAA 83)
- finite semigroups

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First step: read Zádori's paper

Theorem (Thm. 2.3 in Zádori, cat. eq. of finite groups, 1997)

$\mathbf{A}$, $\mathbf{B}$ *two finite algebras.*

$$\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B} \iff \exists \underset{\sim}{\mathbf{A}} \text{ for } \mathbf{A}\, \exists \underset{\sim}{\mathbf{B}} \text{ for } \mathbf{B}\colon\ \mathrm{type}(\underset{\sim}{\mathbf{A}}) = \mathrm{type}(\underset{\sim}{\mathbf{B}}) \wedge$$
$$\text{“}\mathrm{MinRelSets}(\underset{\sim}{\mathbf{A}})/{\cong}\text{”} = \text{“}\mathrm{MinRelSets}(\underset{\sim}{\mathbf{B}})/{\cong}\text{”}$$

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Minimal resets correspond to certain irreducible neighbourhoods (= images of idempotent unary terms of $\mathbf{A}$ / idempotent retracts of $\underset{\sim\sim}{\mathbf{A}}$).

Second step: characterise the irreducible (minimal) neighbourhoods in finite semigroups

Definition (Two sorts of elements in a finite semigroup $\mathbf{A}$)

- $x \in A$ group element $:\Longleftrightarrow \exists n \in \mathbb{N}\colon\ \langle x \rangle_{\mathbf{A}} \cong \mathbb{Z}_n$ group
- $\mathrm{Gr}(\mathbf{A})$ set of all group elements.
- $\mathbf{A}$ completely regular $:\Longleftrightarrow \mathrm{Gr}(\mathbf{A}) = A$.
- For $x \in A$, let $\mathrm{ord}(x) := |\langle x \rangle_{\mathbf{A}}|$.
- $\exp(\mathrm{Gr}(\mathbf{A})) := \mathrm{lcm}\{\mathrm{ord}(x) \mid x \in \mathrm{Gr}(\mathbf{A})\}$

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Proposition (Neighbourhoods of a finite semigroup $\mathbf{A}$)

$\exp(\mathrm{Gr}(\mathbf{A})) = \prod_{i\in I} q_i$ *with distinct prime powers* q_i.

- $\mathbf{A}$ *completely regular*
  $\Longrightarrow\ (\mathrm{Neigh}\,\mathbf{A}, \subseteq) \cong (\mathcal{P}(\{q_i \mid i \in I\}), \subseteq)$
- $\mathbf{A}$ *not completely regular*
  $\Longrightarrow\ (\mathrm{Neigh}\,\mathbf{A}, \subseteq) \cong (\mathcal{P}(\{q_i \mid i \in I\}), \subseteq) + \top$

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- *irreducible: atoms,* $\perp$

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- *minimal: atoms,* $\perp$ *(*$\top$ *if* $\mathbf{A}$ *not completely regular)*

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- *minimal: all pairwise nonisomorphic*

Third step: read Zádori's paper again...

...and generalise to completely regular semigroups

Proposition

$2 \leq |\mathbf{A}| < \infty$ *completely regular semigroup.*
The product of its minimal resets $\underset{\sim}{\mathbf{U}}_1, \ldots, \underset{\sim}{\mathbf{U}}_n$ *has an* $\subseteq$*-minimal idempotent retract* $\underset{\sim}{\mathbf{U}}$ *(neighbourhood) w.r.t. containing a certain set* S*.*

$$S \subseteq \underset{\sim}{\mathbf{U}} \subseteq \prod_{i=1}^{n} \underset{\sim}{\mathbf{U}}_i \twoheadrightarrow \underset{\sim}{\mathbf{U}}$$

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Proposition

$2 \leq |\mathbf{A}| < \infty$ completely regular semigroup. The product of its minimal resets $\underset{\sim}{\mathbf{U}}_1, \ldots, \underset{\sim}{\mathbf{U}}_n$ has an $\subseteq$-minimal idempotent retract $\underset{\sim}{\mathbf{U}}$ (neighbourhood) w.r.t. containing a certain set S.

$$S \subseteq \underset{\sim}{\mathbf{U}} \subseteq \prod_{i=1}^{n} \underset{\sim}{\mathbf{U}}_i \twoheadrightarrow \underset{\sim}{\mathbf{U}} \cong \underset{\sim}{\mathbf{A}}$$

Fourth step: solve the completely regular case

Proposition ($2 \leq |\mathbf{A}|, |\mathbf{B}| < \infty$ completely regular)

$$\mathbf{A} \quad \equiv_{\mathrm{cat}} \quad \mathbf{B}$$

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Proposition ($2 \leq |\mathbf{A}|, |\mathbf{B}| < \infty$ completely regular)

$$
\begin{array}{ccccc}
 & \mathbf{A} & \equiv_{\mathrm{cat}} & \mathbf{B} & \\
 & \rightsquigarrow & & \rightsquigarrow & \\
\textit{reset for } \mathbf{A} & \utilde{\mathbf{A}} & & \utilde{\mathbf{B}} & \textit{reset for } \mathbf{B}
\end{array}
$$

Fourth step: solve the completely regular case

Proposition ($2 \leq |\mathbf{A}|, |\mathbf{B}| < \infty$ completely regular)

$$
\begin{array}{rcccl}
 & \mathbf{A} & \equiv_{\mathrm{cat}} & \mathbf{B} & \\
 & \downarrow & & \downarrow & \\
\textit{reset for } \mathbf{A} & \utilde{\mathbf{A}} & & \utilde{\mathbf{B}} & \textit{reset for } \mathbf{B} \\
\textit{minimal resets} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. &
\begin{array}{c} \utilde{\mathbf{U}}_1 \\ \vdots \\ \utilde{\mathbf{U}}_n \end{array} & &
\begin{array}{c} \utilde{\mathbf{V}}_1 \\ \vdots \\ \utilde{\mathbf{V}}_m \end{array} &
\left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \textit{minimal resets}
\end{array}
$$

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\begin{array}{rccccl}
 & \mathbf{A} & \equiv_{\mathrm{cat}} & \mathbf{B} & \\
 & \rightsquigarrow & & \rightsquigarrow & \\
\textit{reset for } \mathbf{A} & \utilde{\mathbf{A}} & & \utilde{\mathbf{B}} & \textit{reset for } \mathbf{B} \\
\textit{minimal resets} & \left\{\begin{array}{c} \utilde{\mathbf{U}}_1 \\ \vdots \\ \utilde{\mathbf{U}}_n \end{array}\right. & \begin{array}{c} \cong \\ \vdots \\ \cong \end{array} & \left.\begin{array}{c} \utilde{\mathbf{V}}_1 \\ \vdots \\ \utilde{\mathbf{V}}_n \end{array}\right\} & \textit{minimal resets}
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\left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \textit{minimal resets} \\
 & \prod_{i=1}^{n} \utilde{\mathbf{U}}_i & \cong & \prod_{i=1}^{n} \utilde{\mathbf{V}}_i & \\
 & \cup\!\! | & & \cup\!\! | & \\
 & S & & S' & (\exists)
\end{array}
$$

Fourth step: solve the completely regular case

Proposition ($2 \leq |\mathbf{A}|, |\mathbf{B}| < \infty$ completely regular)

$$
\begin{array}{rcccl}
 & \mathbf{A} & \equiv_{\mathrm{cat}} & \mathbf{B} & \\
 & \downarrow & & \downarrow & \\
\textit{reset for } \mathbf{A} & \utilde{\mathbf{A}} & & \utilde{\mathbf{B}} & \textit{reset for } \mathbf{B} \\
\textit{minimal resets}\left\{\vphantom{\begin{array}{c}\utilde{\mathbf{U}}_1\\ \vdots\\ \utilde{\mathbf{U}}_n\end{array}}\right. & \begin{array}{c}\utilde{\mathbf{U}}_1\\ \vdots\\ \utilde{\mathbf{U}}_n\end{array} & \begin{array}{c}\cong\\ \vdots\\ \cong\end{array} & \begin{array}{c}\utilde{\mathbf{V}}_1\\ \vdots\\ \utilde{\mathbf{V}}_n\end{array} & \left.\vphantom{\begin{array}{c}\utilde{\mathbf{V}}_1\\ \vdots\\ \utilde{\mathbf{V}}_n\end{array}}\right\}\textit{minimal resets} \\
 & \prod_{i=1}^{n} \utilde{\mathbf{U}}_i & \cong & \prod_{i=1}^{n} \utilde{\mathbf{V}}_i & \\
 & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & \\
 & S & & S' & (\exists) \\
 & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & \\
 & \utilde{\mathbf{A}} \cong \utilde{\mathbf{U}} & & \utilde{\mathbf{V}} \cong \utilde{\mathbf{B}} & \subseteq\textit{-minimal retract}
\end{array}
$$

Fourth step: solve the completely regular case

Proposition ($2 \le |\mathbf{A}|, |\mathbf{B}| < \infty$ completely regular)

$$
\begin{array}{rcccl}
 & \mathbf{A} & \equiv_{\mathrm{cat}} & \mathbf{B} & \\
 & \big\downarrow\!\wr & & \big\downarrow\!\wr & \\
\textit{reset for } \mathbf{A} & \utilde{\mathbf{A}} & & \utilde{\mathbf{B}} & \textit{reset for } \mathbf{B} \\
\textit{minimal resets} & \left\{\begin{array}{c} \utilde{\mathbf{U}}_1 \\ \vdots \\ \utilde{\mathbf{U}}_n \end{array}\right. & \begin{array}{c} \cong \\ \vdots \\ \cong \end{array} & \left.\begin{array}{c} \utilde{\mathbf{V}}_1 \\ \vdots \\ \utilde{\mathbf{V}}_n \end{array}\right\} & \textit{minimal resets} \\
 & \prod_{i=1}^{n} \utilde{\mathbf{U}}_i & \cong & \prod_{i=1}^{n} \utilde{\mathbf{V}}_i & \\
 & \rotatebox[origin=c]{-90}{$\supseteq$} & & \rotatebox[origin=c]{-90}{$\supseteq$} & \\
 & S & & S' & (\exists) \\
 & \rotatebox[origin=c]{90}{$\supseteq$} & & \rotatebox[origin=c]{90}{$\supseteq$} & \\
 & \utilde{\mathbf{A}} \cong \utilde{\mathbf{U}} & \cong & \utilde{\mathbf{V}} \cong \utilde{\mathbf{B}} & \subseteq\textit{-minimal retract}
\end{array}
$$

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Proposition ($2 \leq |\mathbf{A}|, |\mathbf{B}| < \infty$ completely regular)

$$
\begin{array}{rcccl}
 & \mathbf{A} & \equiv_{\mathrm{cat}} & \mathbf{B} & \\
 & \wr\!\downarrow & & \wr\!\downarrow & \\
\textit{reset for } \mathbf{A} & \utilde{\mathbf{A}} & & \utilde{\mathbf{B}} & \textit{reset for } \mathbf{B}
\end{array}
$$

$$
\textit{minimal resets} \left\{ \begin{array}{ccc}
\utilde{\mathbf{U}}_1 & \cong & \utilde{\mathbf{V}}_1 \\
\vdots & \vdots & \vdots \\
\utilde{\mathbf{U}}_n & \cong & \utilde{\mathbf{V}}_n
\end{array} \right\} \textit{minimal resets}
$$

$$
\begin{array}{rcccl}
 & \prod_{i=1}^{n} \utilde{\mathbf{U}}_i & \cong & \prod_{i=1}^{n} \utilde{\mathbf{V}}_i & \\
 & \cup\!\mid & & \cup\!\mid & \\
 & S & & S' & (\exists) \\
 & \mid\!\cap & & \mid\!\cap & \\
 & \utilde{\mathbf{A}} \cong \utilde{\mathbf{U}} & \cong & \utilde{\mathbf{V}} \cong \utilde{\mathbf{B}} & \subseteq\textit{-minimal retract} \\
 & \Downarrow & & \Downarrow & \\
 & \mathbf{A} & \cong_{\mathrm{w}} & \mathbf{B} &
\end{array}
$$

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Fifth step: do not again read Zádori's paper...

...do something yourself

Proposition ($\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B}$ both non-completely regular)

$$\mathbf{A} \quad \equiv_{\mathrm{cat}} \quad \mathbf{B}$$

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Proposition ($\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B}$ both non-completely regular)

$$\begin{tikzcd}
 & \mathbf{A} \arrow[d, rightsquigarrow] & \equiv_{\mathrm{cat}} & \mathbf{B} \arrow[d, rightsquigarrow] & \\
\textit{reset for } \mathbf{A} & \utilde{\mathbf{A}} & & \utilde{\mathbf{B}} & \textit{reset for } \mathbf{B}
\end{tikzcd}$$

$$\textit{minimal resets}\left\{\begin{array}{c} \utilde{\mathbf{U}}_1 \\ \vdots \\ \utilde{\mathbf{U}}_n \end{array}\right. \qquad\qquad \left.\begin{array}{c} \utilde{\mathbf{V}}_1 \\ \vdots \\ \utilde{\mathbf{V}}_m \end{array}\right\}\textit{minimal resets}$$

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Fifth step: do not again read Zádori's paper…

…do something yourself

Proposition ($\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B}$ both non-completely regular)

$$
\begin{array}{rcccl}
 & \mathbf{A} & \equiv_{\mathrm{cat}} & \mathbf{B} & \\
 & \rightsquigarrow & & \rightsquigarrow & \\
\textit{reset for } \mathbf{A} & \utilde{\mathbf{A}} & & \utilde{\mathbf{B}} & \textit{reset for } \mathbf{B} \\
\textit{minimal resets} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \begin{array}{c} \utilde{\mathbf{U}}_1 \\ \vdots \\ \utilde{\mathbf{U}}_n \end{array} & \begin{array}{c} \cong \\ \vdots \\ \cong \end{array} & \begin{array}{c} \utilde{\mathbf{V}}_1 \\ \vdots \\ \utilde{\mathbf{V}}_n \end{array} & \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \textit{minimal resets}
\end{array}
$$

Fifth step: do not again read Zádori's paper...

...do something yourself

Proposition ($\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B}$ both non-completely regular)

$$\begin{array}{rccl} & \mathbf{A} \quad \equiv_{\mathrm{cat}} \quad \mathbf{B} & \\ & \downarrow \qquad\qquad\quad \downarrow & \\ \textit{reset for } \mathbf{A} & \underset{\sim}{\mathbf{A}} \qquad\qquad\quad \underset{\sim}{\mathbf{B}} & \textit{reset for } \mathbf{B} \end{array}$$

$$\textit{minimal resets} \left\{ \begin{array}{ccc} \underset{\sim}{\mathbf{U}}_1 & \cong & \underset{\sim}{\mathbf{V}}_1 \\ \vdots & \vdots & \vdots \\ \underset{\sim}{\mathbf{A}} & \cong & \underset{\sim}{\mathbf{B}} \end{array} \right\} \textit{minimal resets}$$

Fifth step: do not again read Zádori's paper...

...do something yourself

Proposition ($\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B}$ both non-completely regular)

$$
\begin{array}{ccccc}
 & \mathbf{A} & \equiv_{\mathrm{cat}} & \mathbf{B} & \\
 & \wr\!\downarrow & & \wr\!\downarrow & \\
\textit{reset for } \mathbf{A} & \utilde{\mathbf{A}} & & \utilde{\mathbf{B}} & \textit{reset for } \mathbf{B} \\
\textit{minimal resets} & \left\{\begin{array}{c} \utilde{\mathbf{U}}_1 \\ \vdots \\ \utilde{\mathbf{A}} \end{array}\right. & \begin{array}{c} \cong \\ \vdots \\ \cong \end{array} & \left.\begin{array}{c} \utilde{\mathbf{V}}_1 \\ \vdots \\ \utilde{\mathbf{B}} \end{array}\right\} & \textit{minimal resets} \\
 & \Downarrow & & \Downarrow & \\
 & \mathbf{A} & \cong_{\mathrm{w}} & \mathbf{B} &
\end{array}
$$

Sixth step: solve the mixed case

Proposition

$\mathbf{A}$ *not completely regular,* $\mathbf{B}$ *completely regular*
$\underset{\sim}{\mathbf{A}}$ *reset for* $\mathbf{A}$*;* $\underset{\sim}{\mathbf{B}}$ *reset for* $\mathbf{B}$*.*
$\implies \ldots \implies \exp(\mathrm{Gr}(\mathbf{B})) = 2^n$ *(*$n \geq 1$*) and* $\underset{\sim}{\mathbf{A}} \cong \underset{\sim}{\mathbf{B}}$
$\implies \mathbf{A} \cong_{\mathrm{w}} \mathbf{B}$*.*

Done!

Theorem

For finite semigroups $\mathbf{A}$ *and* $\mathbf{B}$ *we have*

$$\mathbf{A} \equiv_{\mathrm{cat}} \mathbf{B} \iff \mathbf{A} \cong_{\mathrm{w}} \mathbf{B}.$$

Thank you for your attention.