The Valeriote Conjecture

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and
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Celebrating the 80th birthday of Béla Csákány

Result and outline

Conjecture (the Valeriote conjecture, or the Edinburgh conj.)

If a finite algebra $\mathbf{A}$ is finitely related and $\mathrm{HSP}(\mathbf{A})$ is congruence modular, then $\mathbf{A}$ has few subpowers.

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Matt was right.

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Conjecture (the Valeriote conjecture, or the Edinburgh conj.)

If a finite algebra **A** is finitely related and HSP(**A**) is congruence modular, then **A** has few subpowers.

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- (1) Several relevant classes of algebras
- (2) Finitely related algebras
- (3) Collapses of Maltsev conditions, the Valeriote conjecture
- (4) Consequences
- (5) Proof

(1)
Several classes of algebras

CP,CD,CM,FS,NU

$$\begin{tikzcd}
 & & \begin{array}{c}\text{CM}\\ \text{(modular)}\end{array} & \\
 & \begin{array}{c}\text{FS}\\ \text{(few subpowers)}\end{array} \arrow[ur] & & \begin{array}{c}\text{CD}\\ \text{(distributive)}\end{array} \arrow[ul] \\
\begin{array}{c}\text{CP}\\ \text{(permutable)}\end{array} \arrow[ur] & & \begin{array}{c}\text{NU}\\ \text{(near unanimity)}\end{array} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\mathbf{A} \in$ CP/CD/CM if

HSP($\mathbf{A}$) is congruence permutable/distributive/modular

$\mathbf{A} \in$ NU if

$\mathbf{A}$ has a near unanimity term...$f(x, \ldots, x, y, x, \ldots, x) \approx x$

$\mathbf{A} \in$ FS if

$\mathbf{A}$ has few subpowers

Few subpowers

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$\Leftrightarrow$ every $R \leq \mathbf{A}^n$ has a generating set of size $\leq q(n)$, where q is a polynomial

$\Leftrightarrow$ $\mathbf{A}$ satisfies a Maltsev condition of the form

$f(x, ?, ?, \dots, ?) \approx y$
$f(?, x, ?, \dots, ?) \approx y$ where each $? \in \{x, y\}$
$\dots$
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Facts: CP $\Rightarrow$ FS $\Leftarrow$ NU.

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FS + CD $\Rightarrow$ NU (Markovic, McKenzie'08)

(2)
Finitely related algebras

Pol-Inv again

$\mathbb{A}$ ... relational structure on A

$\mathrm{Pol}(\mathbb{A})$ ... clone of all operations compatible with $\mathbb{A}$

Theorem (Geiger, Bodnarčuk, Kalužnin, Kotov, Romov'68)

$\forall$ *finite algebra* $\mathbf{A}$ $\exists \mathbb{A}$ *such that* $\mathrm{Pol}(\mathbb{A}) = \mathrm{Clo}(\mathbf{A})$.

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Definition

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- Algebras with a near unanimity term (by Baker-Pixley)
- Finite relatedness not preserved by H, S, or P Davey, Jackson, Pikethly, Szabó

FS $\Rightarrow$ finitely related

Theorem (Aichinger, Mayr, McKenzie 09)

Every finite algebra with few subpowers is finitely related.

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  (in particular, there is countably many Maltsev clones. This was open even for expansions of $\mathbb{Z}_8$.)

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- **Bonus:** idempotent $\mathbf{A} \in FS$ iff every idempotent expansion is finitely related

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Collapses of Maltsev conditions

Collapses for finite algebras

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- any nontrivial idempotent Maltsev condition (Taylor term) $\Rightarrow$ weak near unanimity term (Maróti, McKenzie 06)
  $f(y, x, \ldots, x) \approx f(x, y, x, \ldots) \approx \cdots \approx f(x, \ldots, x, y)$
- Taylor term $\Rightarrow$ cyclic term (Barto, Kozik 09)
  $f(x_1, \ldots, x_n) \approx f(x_2, \ldots, x_n, x_1)$
- Taylor term $\Rightarrow$ Siggers term (Siggers; Kearnes, Marković, McKenzie 09)
  $f(x, y, z, x) \approx f(y, z, x, z)$
- ...
- Jónsson terms $\Rightarrow$ directed Jónsson terms (Kozik)
- Gumm terms $\Rightarrow$ directed Gumm terms (Kozik)

Collapses for finitely related algebras

Even more collapses of Maltsev conditions for finitely related algebras:

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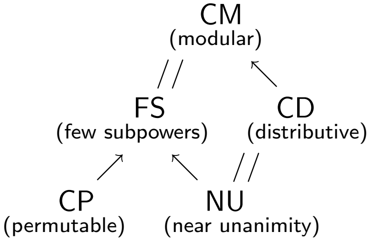
  - Proved Maróti, Zádori'12 for reflexive digraphs

Picture for finitely related algebras



(4)
Consequences

Deciding FS

Corollary

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- More on algebraic problems of this sort in Freese, Valeriote 09

CSP

Corollary

If $\mathbb{A}$ *has finitely many relations and* $\mathsf{Pol}(\mathbb{A}) \in$ *CM then* $\mathrm{CSP}(\mathbb{A})$ *is in P.*

Open problem (!): What if $\mathbb{A}$ has infinitely many relations?

pp-formula comparison

Fix $\mathbb{A}$, $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$

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- **Corollary:** now the P/coNP-complete part is done

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Proof

Audience

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The strategy is

1. find a tree definition of an essential relation of a large arity n

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1. find a tree definition of an essential relation of a large arity n
2. find a comb definition of an essential relation of a (still) large arity (roughly $\log n$)

The strategy

- We have $\mathbb{A}$, $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$
- Want to show $\mathbf{A} \in CM$ (resp. CD) $\Rightarrow \mathbf{A} \in FS$ (resp. NU)
- WLOG $\mathbb{A}$ contains only unary and binary relations
- Assume $\mathbf{A} \in CD$ and $\mathbf{A} \notin NU$
- By Baker-Pixley, $\forall n \;\; \exists R \leq \mathbf{A}^n$ which is essential (= not determined by projections on $n-1$ coordinates)
- By THE Galois correspondence, subpowers of $\mathbf{A}$ can be pp-defined from $\mathbb{A}$
- (pp-definition can be drawn as a labeled digraph [pic])

The strategy is

1. find a tree definition of an essential relation of a large arity n
2. find a comb definition of an essential relation of a (still) large arity (roughly $\log n$)
3. use it to obtain a configuration (some subpowers of $\mathbf{A}$) which contradict $\mathbf{A} \in CD$

More on the strategy for CD $\Rightarrow$ NU

1. Find a tree definition of an essential relation of a large arity

   This can be done in two ways

   - Easy way: Use our result with Kozik about CSPs
   - Harder way: Use Zhuk's technique
2. Find a comb definition of an essential relation of a large arity
   - Easy: Find a long path, take it, shake the tree, fix some elements
3. Obtain an impossible configuration
   - Not too hard
   - Easiest impossible configuration: $\{(c, a, a), (c, b, b), (d, a, b)\}$
   - Directed Jónsson terms (Kozik) simplify the proof of impossibility
4. Celebrate

More on the strategy for CM $\Rightarrow$ FS

1. Find a tree definition of a bad relation of a large arity

   I cannot do it in two ways

   - Easy way: ???????
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   - Directed Gumm terms (Kozik) simplify the proof of impossibility
4. Celebrate more!

My hero (Dmitriy Zhuk)

Bad relations – cube term blockers

Definition

A cube-term blocker in $\mathbf{A}$ is a pair $\mathbf{I} < \mathbf{B} \leq \mathbf{A}$ such that
$\forall t \in \mathrm{Clo}(\mathbf{A})\ \exists i$ so that $t(B, B, \ldots, B, I, B, \ldots, B) \subseteq I$
(I is at the i-th position)

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- Denote $O = B \setminus I$
- $\mathbf{I}, \mathbf{B}$ is a blocker iff $(B^n \setminus O^n) \leq \mathbf{A}^n$ for each n

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Theorem (Marković, Maróti, McKenzie)

$\mathbf{A}$ *finite idempotent. TFAE*

(i) $\mathbf{A} \in FS$

(ii) $\mathbf{A}$ *has no cube-term blockers*

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bad relation $\approx B^n \setminus O^n$ for minimal B

Directed Jónsson terms

Theorem (Jónsson 68)

(Possibly infinite) algebra $\mathbf{A}$ *is in CD iff* $\exists\, p_1, p_2 \dots \in \mathrm{Clo}(\mathbf{A})$

$$\begin{aligned} p_i(x,y,x) &\approx x \\ x &\approx p_1(x,x,y) \\ p_1(x,y,y) &\approx p_2(x,y,y) \\ p_2(x,x,y) &\approx p_3(x,x,y) \\ &\dots \\ p_n(x,y,y) &\approx y \end{aligned}$$

- $F = F(\{x,y\})$, $Q = \langle (x,x,x),(x,y,y),(y,x,y) \rangle \leq F^3$
- $R = \{(b,c) \in Q : \exists\, a\ (a,b,c) \in Q'\}$ (dashed)
- $S = \{(b,c) \in Q : (x,b,c)\}$ (solid)
- solid $\triangleleft_j$ dashed
- Jónsson terms $=$ x, y connected in S

Directed Jónsson terms

Theorem (Kozik)

Finite algebra $\mathbf{A}$ *is in CD iff* $\exists\ p_1, p_2 \dots \in \mathrm{Clo}(\mathbf{A})$

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- directed Jónsson terms = directed path from x to y

Directed Gumm terms

Theorem (Kozik)

Finite algebra $\mathbf{A}$ *in CM iff* $\exists\ p_1, p_2 \dots \in \mathrm{Clo}(\mathbf{A})$

$$
\begin{aligned}
p_i(x,y,x) &\approx x \\
x &\approx p_1(x,x,y) \\
p_1(x,y,y) &\approx p_2(x,x,y) \\
p_2(x,y,y) &\approx p_3(x,x,y) \\
&\dots \\
p_n(x,y,y) &\approx q(x,y,y) \\
q(x,x,y) &\approx y
\end{aligned}
$$

Question: Jónsson (resp. Gumm) terms $\Rightarrow$ directed Jónsson (resp. Gumm) terms for infinite algebras?

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- Start with a pp-definition

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  - Exercise in graph theory, see Zhuk or Maróti, Zádori

Jónsson and Gumm absorption

Jónsson absorption:

- vertex absorption preserves connectivity
- edge absorption preserves connectivity

Gumm absorption:

- vertex absorption preserves connectivity
- edge absorption does not, but q is Maltsev modulo components

Thank you!