On the tolerance lattice of tolerance factors

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Joint work with Sándor Radeleczki[1]

A tolerance relation of a lattice is a reflexive and symmetric relation compatible with operations of the lattice. All tolerances of a lattice L, denoted by $\mathrm{Tol}(L)$, form a lattice (with respect to inclusion). A block of a tolerance $T \in \mathrm{Tol}(L)$ is such a maximal subset B of L that $B^2 \subseteq T$. Blocks are convex sublattices of L and it was shown by G. Czédli that they form a lattice (denoted by L/T) called the factor lattice of L modulo T.

Although the notion of a tolerance is a natural generalization of the notion of a congruence, many properties of factor lattices modulo congruences are not, in general, valid for factor lattices modulo tolerances. For instance, it is well known that in the case of a congruence $\varphi \in \mathrm{Con}(L)$ the congruence lattice of the factor lattice L/φ is isomorphic to the principal filter $[\varphi)$ in $\mathrm{Con}(L)$ (the homomorphism theorem). Moreover, every $\psi \in \mathrm{Con}(L)$ such that $\psi \geq \varphi$ induces the congruence ψ/φ on the factor lattice L/φ such that $(L/\varphi)/(\psi/\varphi) \cong L/\psi$ (the second isomorphism theorem). In this paper, for a finite lattice L, we define a new partial order $\sqsubseteq$ on $\mathrm{Tol}(L)$ such that for every $S \in \mathrm{Tol}(L)$ with $T \sqsubseteq S$, a tolerance S/T is induced on the factor lattice L/T. This partial order is a particular restriction of $\subseteq$ and by means of it we can prove analogous results to the afore mentioned ones.

Albeit the poset $(\mathrm{Tol}(L), \sqsubseteq)$ is not a lattice in general, it has the structure of a specific commutative join-directoid (see, e.g., [3]). Moreover, for every $T \in \mathrm{Tol}(L)$, $(\mathrm{Tol}(L/T), \sqsubseteq)$ constitutes a subdirectoid of the directoid based on the poset $(\mathrm{Tol}(L), \sqsubseteq)$ and this specific directoid structure is preserved by the direct product of lattices.

Part I of this talk will be presented by Joanna Grygiel, Part II by Sándor Radeleczki.

References

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