About the Minkowski problem

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Reconstruction of smooth closed convex surfaces from Gauss curvature

- X is a compact C^2_+ hypersurface in $\mathbb{R}^n$
- u_x is exterior unit normal at $x \in X$
- $\kappa_X(u_x) > 0$ is the Gauss curvature

Observation (Minkowski)

$$\int_{S^{n-1}} u \cdot \kappa_X(u)^{-1}\, du = o. \tag{1}$$

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Minkowski problem (E.g. Inverse problem of short wave diffraction)
For continuous $\kappa : S^{n-1} \to \mathbb{R}_+$ satisfying (1), find C^2_+ hypersurface $X \subset \mathbb{R}^n$ such that $\kappa(u_x)$ is the Gauss curvature at $x \in X$.

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Monge-Ampere type differential equation on S^{n-1}:

$$\det(\nabla^2 h + h\, I) = \kappa^{-1}$$

where $h(u) = \max\{\langle u, x\rangle : x \in X\}$ is the support function.

Notation

- K, C - convex bodies in $\mathbb{R}^n$
  (convex compact with non-empty interior)
- $V(K)$ - volume (Lebesgue measure)
- $\mathcal{H}^{n-1}$ - $(n-1)$-Hausdorff measure
- h_K - support function of K
  $h_K(u) = \max\{\langle u, x\rangle : x \in K\}$ for $u \in \mathbb{R}^n$
- L - linear subspace, $L \neq \{o\}, \mathbb{R}^n$
- μ - non-trivial Borel measure on S^{n-1}

Surface area measure

S_K - surface area measure of K on S^{n-1}

- $\nu_K(x) = \{u \in S^{n-1} : h_K(u) = \langle x, u \rangle\}$ for $x \in \partial K$ (all possible exterior unit normals at x)
- For $\Xi \subset \partial K$, $S_K(\nu_K(\Xi)) = \mathcal{H}^{n-1}(\Xi)$

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- K polytope, $F_1, \ldots, F_k$ facets, u_i exterior unit normal at F_i

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"Minkowski problem" : Given μ, find K with $\mu = S_K$
Solution (Minkowski, Alexandrov, Nirenberg)

$$\int_{S^{n-1}} u \, dS_K(u) = o$$

- Minimize $\int_{S^{n-1}} h_C \, d\mu$ under the condition $V(C) = 1$

L_p surface area measures

L_p surface area measures (Firey, Lutwak 1990) $p \in \mathbb{R}$

$$dS_{K,p} = h_K^{1-p}\, dS_K$$

Examples

- $S_{K,1} = S_K$
- $S_{K,0}$ "cone-volume meaure"
- $S_{K,-n}$ related to the $\mathrm{SL}(n)$ invariant curvature $\frac{\kappa_K(u)}{h_K(u)^{n+1}}$

Theorem (Chou-Wang (2005), Hug-LYZ (2006))

If $p > 1$, $p \neq n$, then any finite Borel measure μ on S^{n-1} not concentrated on any closed hemisphere is of the form $\mu = S_{K,p}$.

Remark Possibly $o \in \partial K$ if $1 < p < n$

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Theorem (Zhu (2015))

If $p < 1$, then any "general" discrete measure μ on S^{n-1} not concentrated on any closed hemisphere is of the form $\mu = S_{K,p}$.

Differential equation for L_p surface area measures

$$h^{1-p}\det(\nabla^2 h + hI) = f$$

Theorem

There is a solution K with $o \in K$ and $f\, d\mathcal{H}^{n-1} = dS_{K,p}$ provided

- $0 \leq p < 1$ *and f is in L^1* *(Chen, Li, Zhu)*
- $-n < p < 0$ *and f is in $L^{\frac{n}{n+p}}$* *(Bianchi, B, Colesanti)*

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Theorem

$0 < \inf f \le \sup f < \infty$

- *If $p \le 2 - n$, then $o \in \mathrm{int} K$, and hence K is smooth and strictly convex* (Chou, Wang)
- *If $p \le 4 - n$, then K is smooth* (Bianchi, B, Colesanti)
- *If $o \in \mathrm{int} K$ and f is C^α, then ∂K is $C^{2,\alpha}$* (Caffarelli)

Ideas to solve L_p-Minkowski problem for given μ

$p > 1$

- Minimize $\int_{S^{n-1}} h_K^p \, d\mu$ under the condition $V(K) = 1$
- Weak approximation by discrete measures (polytopes)

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$p < 1$

$$\varphi(t) = \begin{cases} t^p & = & \text{if } 0 < p < 1 \\ \log t & = & \text{if } p = 0 \\ -t^p & = & \text{if } p < 0 \end{cases}$$

For $\mathcal{K} = \{$convex body K: $o \in K$ and $V(K) = 1\}$, find

$$\inf_{K \in \mathcal{K}} \sup_{\xi \in \operatorname{int} K} \int_{S^{d-1}} \varphi \circ h_{K-\xi} \, d\mu$$

Dual curvature measures

Dual curvature measures

Huang, Lutwak, Yang, Zhang, 2016 (Acta Mathematica)

$$\widetilde{C}_{K,q}(\nu_K \circ r_K(\omega)) = \int_\omega \varrho_K^q(u)\, du \quad \text{for } \omega \subset S^{n-1}$$

$u \in S^{n-1} \Longrightarrow r_K(u) = \varrho_K(u)u \in \partial K,\ \varrho_K(u) \geq 0$

Example

- $\widetilde{C}_{K,n} = S_{0,K} \iff d\widetilde{C}_{K,n} = h_K dS_K$

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Theorem (Zhao (2016), B-Henk-Pollehn (2016))

$0 < q < n$, and μ is finite even non-trival Borel measure on S^{n-1}. Then $\mu = \widetilde{C}_{K,q}$ for o-symmetric K iff for every non-trivial L,

$$\mu(L \cap S^{n-1}) < \frac{\dim L}{q} \cdot \mu(S^{n-1}).$$

L_p dual curvature measures

Lutwak, Yang, Zhang (2016) $p, q \in \mathbb{R}$

$$d\widetilde{C}_{K,p,q} = h_K^{-p}\, d\widetilde{C}_{K,q}$$

Examples

- $\widetilde{C}_{K,p,n} = S_{K,p}$
- $\widetilde{C}_{K,0,q} = \widetilde{C}_{K,q}$

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Theorem

$\mu = \widetilde{C}_{K,p,n} \iff \mu$ *is not contrated on a great subsphere and*

- $p > 1$ *and* $q > 0$, $p \neq q$ *(B, Fodor)*
- $p > 0$ *and* $q < 0$ *(Huang, Zhao)*

Idea to solve L_p-dual Minkowski problem for $p > 1$, $q > 0$, μ

Minimize $\int_{S^{n-1}} h_K^p\, d\mu$ under the condition $V_q(K) = \frac{1}{n} \int_{S^{n-1}} \varrho_K^q = 1$