About the Minkowski problem

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Szeged, October, 2017

Honoring the Department of Geometry, Szeged
Reconstruction of smooth closed convex surfaces from Gauss curvature

- $X$ is a compact $C^2_+$ hypersurface in $\mathbb{R}^n$
- $u_x$ is exterior unit normal at $x \in X$
- $\kappa_X(u_x) > 0$ is the Gauss curvature

Observation (Minkowski)

$$\int_{S^{n-1}} u \cdot \kappa_X(u)^{-1} \, du = o. \quad (1)$$
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Minkowski problem (E.g. Inverse problem of short wave diffraction)

For continuous $\kappa : S^{n-1} \to \mathbb{R}_+$ satisfying (1), find $C^2_+$ hypersurface $X \subset \mathbb{R}^n$ such that $\kappa(u_x)$ is the Gauss curvature at $x \in X$. 
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Monge-Ampere type differential equation on $S^{n-1}$:

$$\det(\nabla^2 h + h I) = \kappa^{-1}$$

where $h(u) = \max\{\langle u, x \rangle : x \in X\}$ is the support function.
Notation

- $K, C$ - convex bodies in $\mathbb{R}^n$ (convex compact with non-empty interior)
- $V(K)$ - volume (Lebesgue measure)
- $\mathcal{H}^{n-1}$ - $(n - 1)$-Hausdorff measure
- $h_K$ - support function of $K$
  \[ h_K(u) = \max\{\langle u, x \rangle : x \in K\} \text{ for } u \in \mathbb{R}^n \]
- $L$ - linear subspace, $L \neq \{o\}, \mathbb{R}^n$
- $\mu$ - non-trivial Borel measure on $S^{n-1}$
Surface area measure

$S_K$ - surface area measure of $K$ on $S^{n-1}$

$\nu_K(x) = \{u \in S^{n-1} : h_K(u) = \langle x, u \rangle \}$ for $x \in \partial K$
(all possible exterior unit normals at $x$)

For $\Xi \subset \partial K$, $S_K(\nu_K(\Xi)) = \mathcal{H}^{n-1}(\Xi)$
Surface area measure

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\( K \) polytope, \( F_1, \ldots, F_k \) facets, \( u_i \) exterior unit normal at \( F_i \)

\[ S_K(\{u_i\}) = \mathcal{H}^{n-1}(F_i). \]
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- $K$ polytope, $F_1, \ldots, F_k$ facets, $u_i$ exterior unit normal at $F_i$
  
  $$S_K(\{u_i\}) = \mathcal{H}^{n-1}(F_i).$$

"Minkowski problem" : Given $\mu$, find $K$ with $\mu = S_K$

Solution (Minkowski, Alexandrov, Nirenberg)

$$\int_{S^{n-1}} u \, dS_K(u) = 0$$

- Minimize $\int_{S^{n-1}} h_C \, d\mu$ under the condition $V(C) = 1$
$L_p$ surface area measures

$L_p$ surface area measures (Firey, Lutwak 1990) $p \in \mathbb{R}$

$$dS_{K,p} = h_K^{1-p} dS_K$$

Examples

- $S_{K,1} = S_K$
- $S_{K,0}$ "cone-volume measure"
- $S_{K,-n}$ related to the $\text{SL}(n)$ invariant curvature $\frac{\kappa_K(u)}{h_K(u)^{n+1}}$

Theorem (Chou-Wang (2005), Hug-LYZ (2006))

If $p > 1$, $p \neq n$, then any finite Borel measure $\mu$ on $S^{n-1}$ not concentrated on any closed hemisphere is of the form $\mu = S_{K,p}$.

Remark  Possibly $o \in \partial K$ if $1 < p < n$
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Theorem (Zhu (2015))

If $p < 1$, then any ”general” discrete measure $\mu$ on $S^{n-1}$ not concentrated on any closed hemisphere is of the form $\mu = S_{K,p}$.
Differential equation for $L_p$ surface area measures

$$h^{1-p} \det(\nabla^2 h + hI) = f$$

**Theorem**

There is a solution $K$ with $o \in K$ and $f \, d\mathcal{H}^{n-1} = dS_{K,p}$ provided

- $0 \leq p < 1$ and $f$ is in $L^1$ \textit{(Chen, Li, Zhu)}
- $-n < p < 0$ and $f$ is in $L^{\frac{n}{n+p}}$ \textit{(Bianchi, B, Colesanti)}
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**Theorem**

$0 < \inf f \leq \sup f < \infty$

- If $p \leq 2 - n$, then $o \in \text{int}K$, and hence $K$ is smooth and strictly convex (Chou, Wang)
- If $p \leq 4 - n$, then $K$ is smooth (Bianchi, B, Colesanti)
- If $o \in \text{int}K$ and $f$ is $C^\alpha$, then $\partial K$ is $C^{2,\alpha}$ (Caffarelli)
Ideas to solve $L_p$-Minkowski problem for given $\mu$

$p > 1$

- Minimize $\int_{S^{n-1}} h_K^p \, d\mu$ under the condition $V(K) = 1$
- Weak approximation by discrete measures (polytopes)
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$p < 1$

$$\varphi(t) = \begin{cases} 
  t^p & \text{if } 0 < p < 1 \\
  \log t & \text{if } p = 0 \\
  -t^p & \text{if } p < 0 
\end{cases}$$

For $\mathcal{K} = \{\text{convex body } K: \ o \in K \text{ and } V(K) = 1\}$, find

$$\inf_{K \in \mathcal{K}} \sup_{\xi \in \text{int} K} \int_{S^{d-1}} \varphi \circ h_{K-\xi} \, d\mu$$
Dual curvature measures

Huang, Lutwak, Yang, Zhang, 2016 (Acta Mathematica)

\[ \tilde{C}_{K,q}(\nu_K \circ r_K(\omega)) = \int_{\omega} \varrho^q_K(u) \, du \quad \text{for } \omega \subset S^{n-1} \]

\[ u \in S^{n-1} \implies r_K(u) = \varrho_K(u)u \in \partial K, \quad \varrho_K(u) \geq 0 \]

Example

\[ \tilde{C}_{K,n} = S_{0,K} \iff d\tilde{C}_{K,n} = h_K dS_K \]
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\[ u \in S^{n-1} \implies r_K (u) = \varphi_K (u) \quad u \in \partial K, \; \varphi_K (u) \geq 0 \]

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Theorem (Zhao (2016), B-Henk-Pollehn (2016))

0 < q < n, and \( \mu \) is finite even non-trivial Borel measure on \( S^{n-1} \). Then \( \mu = \tilde{C}_{K,q} \) for \( o \)-symmetric \( K \) iff for every non-trivial \( L \),

\[ \mu (L \cap S^{n-1}) < \frac{\dim L}{q} \cdot \mu (S^{n-1}). \]
$L_p$ dual curvature measures

Lutwak, Yang, Zhang (2016) $p, q \in \mathbb{R}$

$$d\tilde{C}_{K,p,q} = h_K^{-p} d\tilde{C}_{K,q}$$

Examples

- $\tilde{C}_{K,p,n} = S_{K,p}$
- $\tilde{C}_{K,0,q} = \tilde{C}_{K,q}$
\( L_p \) dual curvature measures

Lutwak, Yang, Zhang (2016) \( p, q \in \mathbb{R} \)

\[
d\tilde{C}_{K,p,q} = h_{K}^{-p} \, d\tilde{C}_{K,q}
\]

Examples

\[ \tilde{C}_{K,p,n} = S_{K,p} \]

\[ \tilde{C}_{K,0,q} = \tilde{C}_{K,q} \]

Theorem

\( \mu = \tilde{C}_{K,p,n} \iff \mu \) is not contrated on a great subsphere and

\[ p > 1 \text{ and } q > 0, \ p \neq q \ (B, \ Fodor) \]

\[ p > 0 \text{ and } q < 0 \ (Huang, \ Zhao) \]

Idea to solve \( L_p \)-dual Minkowski problem for \( p > 1, \ q > 0, \ \mu \)

Minimize \( \int_{S^{n-1}} h_K^p \, d\mu \) under the condition \( V_q(K) = \frac{1}{n} \int_{S^{n-1}} q^q_K = 1 \)