On chromatic indices of affine spaces

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Graph decomposition

Definition

A decomposition of a simple graph $G = (V(G), E(G))$ *is a pair* $[G, \mathcal{D}]$ *where* $\mathcal{D}$ *is a set of induced subgraphs of* G*, such that every edge of* G *belongs to exactly one subgraph in* $\mathcal{D}$*.*

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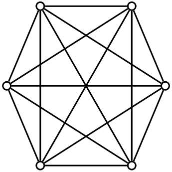
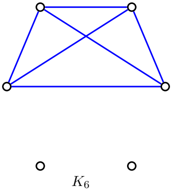


K_6

Graph decomposition



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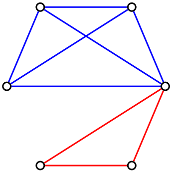

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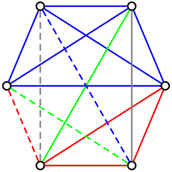
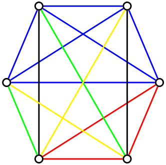


K_6

Coloring

Definition

A coloring of a decomposition $[G, \mathcal{D}]$ *with* k *colors is a surjective function that assigns to edges of* G *a color from a* k*-set of colors, such that all edges of* $H \in \mathcal{D}$ *have the same color. A coloring of* $[G, \mathcal{D}]$ *with* k *colors is proper, if for all* $H_1, H_2 \in \mathcal{D}$ *with* $H_1 \neq H_2$ *and* $V(H_1) \cap V(H_2) \neq \emptyset$*, then* $E(H_1)$ *and* $E(H_2)$ *have different colors.*



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The chromatic index $\chi'([G,\mathcal{D}])$ *of a decomposition is the smallest number* k *for which there exists a proper coloring of* $[G,\mathcal{D}]$ *with* k *colors.*

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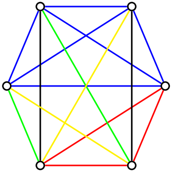

$\chi'(K_6, \mathcal{D}) = 5$

Coloring

Definition

A coloring of $[G, \mathcal{D}]$ *with* k *colors is complete if each pair of colors appears on at least a vertex of* G*. The pseudoachromatic index* $\psi'([G, \mathcal{D}])$ *of a decomposition is the largest number* k *for which there exist a complete coloring with* k *colors.*

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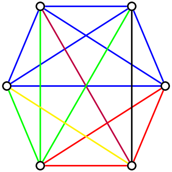

$\psi'(K_6, \mathcal{D}) = 6$

Coloring

Definition

The achromatic index $\alpha'([G,\mathcal{D}])$ *of a decomposition is the largest number* k *for which there exist a proper and complete coloring with* k *colors.*

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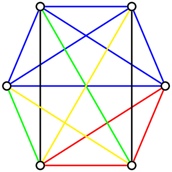

$\alpha'(K_6, \mathcal{D}) = 5$

Coloring

If $\mathcal{D} = E(G)$ then $\chi'([G,E])$, $\alpha'([G,E])$ and $\psi'([G,E])$ are the usual *chromatic*, *achromatic* and *pseudoachromatic indices* of G, respectively.

Coloring

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Clearly we have that

$$\chi'([G, \mathcal{D}]) \leq \alpha'([G, \mathcal{D}]) \leq \psi'([G, \mathcal{D}]).$$

The Erdős-Faber-Lovász Conjecture

Conjecture

For any decomposition $\mathcal{D}$ of K_v, given by complete graphs, satisfies the inequality

$$\chi'([K_v, \mathcal{D}]) \leq v.$$

Designs define decompositions of the corresponding complete graphs in the natural way. Identify the points of a (v, κ)-design $D = (\mathcal{V}, \mathcal{B})$ with the set of vertices of the complete graph K_v. Then the set of points of each block of D induces in K_v a subgraph isomorphic to K_κ and these subgraphs give a decomposition of K_v.

Known results

The EFL Conjecture is open even for the $(v, 3)$–designs (Steiner triple systems $STS(v)$).

Theorem (Colbourn, C. J. – Colbourn, M. J.)

If $\mathcal{D}$ is a (v, κ)–design, then

$$\chi'(\mathcal{D}) < \frac{\kappa v}{\kappa - 1}.$$

Theorem (Colbourn, C. J. – Colbourn, M. J.)

If $\mathcal{D} = (\mathbb{Z}_v, \mathcal{B})$ is a cyclic designs (that is the mapping $i \mapsto i + 1$ is an automorphism), then

$$\chi'(\mathcal{D}) \leq v,$$

the EFL Conjecture is true for cyclic designs.

Known results

The finite projective space $\mathrm{PG}(n, q)$ can be regarded as a $(\frac{q^{n+1}-1}{q-1}, q+1)$–design, where the set of blocks are the set of lines of $\mathrm{PG}(n, q)$

Theorem (Beutelspacher, A. – Jungnickel, D. – Vanstone, S.A.)

If $\mathcal{D}$ is the n-dimensional finite projective space, then

$$\chi'(\mathcal{D}) \leq v,$$

the EFL Conjecture is true for finite projective spaces.

Let Π_q be any finite projective plane of order q. Then $v = q^2 + q + 1$ is the number of points in Π_q. It is not hard to see that

$$\chi'(\Pi_q) = \alpha'(\Pi_q) = \psi'(\Pi_q) = v.$$

Achromatic index

Theorem (Bouchet, A)

If q is an odd number and $v = q^2 + q + 1$ then a projective plane of order q exists if and only if $\alpha'([K_v, E]) = qv$.

As a corollary of this theorem we get that $\alpha'(K_v)$ grows asymtotically, like $v^{3/2}$.

Achromatic index

The achromatic index of $STS(v)$ has been studied before.

Theorem (Colbourn, C. J. – Colbourn, M. J.)

For any $STS(v)$, $\alpha'(STS(v)) \leq cv^{3/2}$ for c is a fixed constant.

Theorem (Colbourn, C. J. – Colbourn, M. J.)

For infinitely many v, there exists an $STS(v)$, such that

$$\alpha'(STS(v)) \geq c'v^{3/2},$$

for some fixed constant c'.

Achromatic index

Theorem (A-P, K, R-M, V-A)

$$\alpha'(\mathrm{PG}(5,q)) \geq c\frac{v^{1.5}}{\kappa-1},\ \textit{where } v=\frac{q^6-1}{q-1},\ \textit{and } c \textit{ a fixed constant}$$

Pseudoachromatic index

Theorem (A-P, K, R-M, V-A)

Let $\mathcal{D}$ be a (v, κ)–design. Then

$$\psi'(\mathcal{D}) \leq \frac{\sqrt{v}(v-1)}{\kappa - 1} < \frac{v^{1.5}}{\kappa - 1}.$$

In the case $\kappa = 3$ this theorem improves Theorem 5.

The finite affine space $\mathrm{AG}(n, q)$ can be regarded as a (q^n, q)–design, where the set of blocks are the set of lines of $\mathrm{AG}(n, q)$

Affine spaces, chromatic index

The finite affine space $\mathrm{AG}(n, q)$ can be regarded as a (q^n, q)–design, where the set of blocks are the set of lines of $\mathrm{AG}(n, q)$

It is not hard to see that

- $\chi'(\mathrm{AG}(n, q)) = \frac{q^n-1}{q-1} < v$,
- $\alpha'(\mathrm{A}_q) = q + 1$, if A_q is any affine plane of order q.

Affine planes

Theorem (A-P, K, R-M, V-A)

Let A_q be any affine plane of order q. Then

$$\psi'(\mathrm{A}_q) = \left\lfloor \frac{(q+1)^2}{2} \right\rfloor .$$

Affine planes

Theorem (A-P, K, R-M, V-A)

Let A_q be any affine plane of order q. Then

$$\psi'(\mathrm{A}_q) = \left\lfloor \frac{(q+1)^2}{2} \right\rfloor.$$

The upper estimate follows from the pigeonhole principle, the lower estimate is an easy constructions.

Theorem (A-P, K, R-M, V-A)

Let $\mathrm{AG}(3,q)$ *be the 3-dimensional affine space of order* q. *Then*

- $\frac{(q^2+q)(q+1)+2}{2} \leq \alpha'(\mathrm{AG}(3,q)) \leq \left\lfloor (q^3+q^2+q)\sqrt{q} - \frac{1}{2}q^3 \right\rfloor$,
- $q^3+1 \leq \psi'(\mathrm{AG}(3,q)) \leq \left\lfloor (q^3+q^2+q)\sqrt{q} - \frac{1}{2}q^3 \right\rfloor$.

3-dimensional affine space

Theorem (A-P, K, R-M, V-A)

Let $\mathrm{AG}(3,q)$ *be the 3-dimensional affine space of order* q*. Then*

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- $q^3+1 \leq \psi'(\mathrm{AG}(3,q)) \leq \left\lfloor (q^3+q^2+q)\sqrt{q} - \frac{1}{2}q^3 \right\rfloor$.

The upper estimate follows from a refinement of Theorem 7, the lower estimates are a bit difficult constructions.

Higher dimensional affine spaces

Theorem (A-P, K, R-M, V-A)

Let $\mathrm{AG}(4,q)$ *be the 4-dimensional affine space of order* q. *Then*

$$\frac{(q^2+1)(q^3+q^2+q)}{2} \le \psi'(\mathrm{AG}(4,q)) \le \left\lfloor \frac{q^6\sqrt{q}}{(q-1)\sqrt{q-1}} \right\rfloor .$$

Higher dimensional affine spaces

Theorem (A-P, K, R-M, V-A)

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Conjecture

Let $\mathrm{AG}(n,q)$ *be the n-dimensional affine space of order* q. *Then*

$$\psi'(\mathrm{AG}(n,q)) \approx q^{n+1}.$$