On Some Properties of Harmonic Spaces

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May 22, 2014

Kneser–Poulsen Conjecture (1954-55)

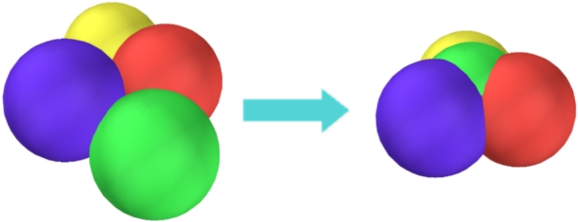
Suppose that for the points $P_1, \dots, P_N$ and $Q_1, \dots, Q_N$ in $\mathbf{E}^n$,

$$d(P_i, P_j) \geq d(Q_i, Q_j) \quad \forall\, 1 \leq i < j \leq N.$$

Do these inequalities imply for the inequality

$$\mathrm{Vol}_n\left(\bigcup_{i=1}^{N} B(P_i, r)\right) \geq \mathrm{Vol}_n\left(\bigcup_{i=1}^{N} B(Q_i, r)\right)$$

for $r > 0$?



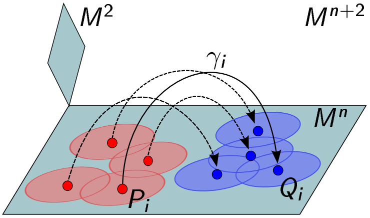
State of the Art

Theorem (K. Bezdek, R. Connelly for $M^n = \mathbf{E}^n$, B. Cs. for $M^n = \mathbf{S}^n$ or $\mathbf{H}^n$)

If $P_i, Q_i \in M^n \subset M^{n+2}$ for $i = 1, \dots, N$, and there are continuous curves $\gamma_i : [0,1] \to M^{n+2}$ such that $\gamma_i(0) = P_i$, $\gamma_i(1) = Q_i$ and $d(\gamma_i(t), \gamma_j(t))$ weakly decreasing for all $1 \le i, j \le N$, then

$$\mathrm{Vol}_n\left(\bigcup_{i=1}^{N} B^n(P_i, r_i)\right) \ge \mathrm{Vol}_n\left(\bigcup_{i=1}^{N} B^n(Q_i, r_i)\right)$$

for all $r_1, \dots, r_N > 0$.



Main tools of the proof:

- a Schläfli-type formula valid in Einstein manifolds;
- an "Archimedean" formula for bodies of revolution in constant curvature spaces.

Question

Can the Kneser-Poulsen conjecture be true in more general spaces?

Definition

We say that a metric space with measure has the property KP_k if the volume of the intersection of k open balls depends only on the radii of the balls and the distances between the centers.

We can introduce a weaker condition according to the original conjecture.

Definition

We say that a metric space with measure has the property $KP_k^=$ if the volume of the intersection of k open balls with equal radii depends only on the common radius of balls and the distances between the centers.

It is obvious that

$$\begin{array}{ccccc}
\begin{array}{c}\text{Kneser-Poulsen}\\ \text{for } k \text{ balls with different radii}\end{array} & \Rightarrow & KP_k & \Rightarrow & KP_{k-1} \\
 & & \Downarrow & & \Downarrow \\
\begin{array}{c}\text{Kneser-Poulsen}\\ \text{for } k \text{ balls with equal radii}\end{array} & \Rightarrow & KP_k^= & \Rightarrow & KP_{k-1}^=
\end{array}$$

- KP_1 and $KP_1^=$ properties are the same.
- KP_1 property is closely related to ball homogeneity introduced by O. Kowalski and L. Vanhecke.
- A Riemannian manifold is called ball homogeneous, if the volume of "small" geodesic balls depends only on the radius of the balls.
- Since

$$\mathrm{Vol}_n(B(p,r)) = V_0^n(r)\left(1 - \frac{s(p)}{6(n+2)}r^2 + O(r^4)\right),$$

ball homogeneous spaces have constant scalar curvature.

Harmonic Manifolds

Definition

A Riemannian manifold is a harmonic space if the followig equivalent definitions are fulfilled.

- Each point $p \in M$ has a normal neighborhood on which the equation $\Delta u = 0$ has a non-constant solution of the form $u(q) = f(d(p,q))$, where $f : [0,a) \to \mathbb{R}$ is real analytic on $(0,a)$. [Ruse 1930]
- At each point $p \in M$, the volume density function $\theta_p = \sqrt{\det(g_{ij})}$ written in normal coordinates centered at p is a radial (spherically symmetric) function.
- Every sufficiently small geodesic sphere has constant mean curvature.
- (if $\dim M > 2$) Every sufficiently small geodesic sphere has constant scalar curvature.
- If f is harmonic on a neighborhood of a sufficiently small geodesic ball $B(p,r)$, then

$$f(p) = \frac{1}{vol(S(p,r))} \int_{S(p,r)} f \,\mathrm{d}\sigma$$

Examples of Harmonic Spaces

2 point homogeneous and rank 1 symmetric spaces

Definition

A metric space (X, d) is called 2 point homogeneous, if for any $p, q, p', q' \in X$ such that $d(p, q) = d(p', q')$, there is an isometry $\Phi : X \to X$ such that $\Phi(p) = p'$ and $\Phi(q) = q'$.

Proposition

A connected Riemannian manifold is 2 point homogeneous if and only if it is complete and its isometry group acts transitively on the bundle of unit tangent vectors. In particular, spaces locally isometric to a 2 point homogeneous space are harmonic.

Theorem (J. Tits, H.C. Wang, Z. Szabó)

Connected 2 point homogeneous Riemannian manifolds are the Euclidean space $\mathbf{E}^n$, the simply connected rank 1 symmetric spaces and $\mathbb{R}\mathbf{P}^n$.*
** [the compact spaces $\mathbf{S}^n$, $\mathbb{C}\mathbf{P}^n$, $\mathbb{H}\mathbf{P}^n$, $\mathbb{C}a\mathbf{P}^2$, and their non-compact duals $\mathbf{H}^n$, $\mathbb{C}\mathbf{H}^n$, $\mathbb{H}\mathbf{H}^n$ and $\mathbb{C}a\mathbf{H}^2$]*

Lichnerowicz Conjecture

Every harmonic space is locally isometric to a rank 1 *symmetric space.*

Szabó's theorem gives a verification of the conjecture in the compact case.

Theorem (Z. Szabó)

If a simply connected connected harmonic space is compact, then it is a rank 1 *symmetric space.*

Remarks

- In the non-compact case, the Lichnerowicz conjecture is false.
- Damek and Ricci observed, that among factor spaces of Heisenberg type 2-step nilpotent Lie groups equipped with left invariant Riemannian metrics, there are many harmonic spaces, which are not symmetric.
- Riemannian spaces locally isometric to a rank 1 symmetric space or a Damek Ricci space are the only known examples of harmonic spaces.

Harmonic manifolds and the KP_2 property

Definition

A kernel function on a set X is a map $X \times X \to \mathbb{R}$. A kernel function F on a metric space (X, d) is said to be radial if there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $F(p, q) = f(d(p, q))$.

Definition

If F and G are kernel functions on a measurable space (X, μ) such that for any $p, q \in X$ the functions $x \mapsto F(p, x)$ and $x \mapsto G(x, p)$ are in $L^2(X, \mu)$, then the convolution of F and G is the kernel function defined by

$$F * G(p, q) = \int_X F(p, x)G(x, q)d\mu(x).$$

Theorem (Z.I. Szabó, 1990)

A simply connected, complete Riemannian manifold is harmonic, if and only if the convolution of radial kernel functions is radial.

Harmonic manifolds and the KP_2 property

Corollary

A simply connected, complete Riemannian manifold is harmonic, if and only if the KP_2 property

Proof.

($\Longrightarrow$) Let $\chi_r : \mathbb{R}_+ \to \mathbb{R}$ denote the characteristic function of the interval $[0, r]$ and set $F_r = \chi_r \circ d$. F_r is a radial kernel function so if M is harmonic, then

$$F_{r_1} * F_{r_2}(p, q) = \int_M \chi_{r_1}(d(p, x))\chi_{r_2}(d(x, q)) = vol_n(B(x, r_1) \cap B(q, r_2))$$

is radial as well so M is KP_2.

($\Longleftarrow$) If M is KP_2, then the convolutions of radial functions of the form $\sum_i c_i F_{r_i} = (\sum_i c_i \chi_{r_i}) \circ d$ are radial, and the set of step functions is dense in the space of compactly supported smooth functions on $\mathbb{R}_+$. □

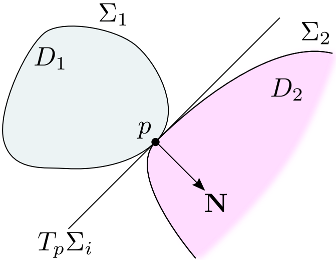
Observation. The second part of the proof needs the KP_2 condition for balls with different radii.

Harmonic manifolds and the $KP_2^=$ property

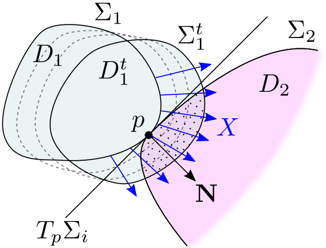
Theorem (B. Cs., M. Horváth)

A simply connected, complete Riemannian manifold is harmonic if and only if it has the $KP_2^=$ property.

Main tool: asymptotical formula for the volume of the intersection of two slightly intersecting balls.



- Let D_1 and D_2 be two regular domains in an n-dimensional Riemannian manifold M, which are tangent to one another at their unique common point p. Suppose that D_1 is compact.
- Denote the boundary of D_1 and D_2 by Σ_1 and Σ_2 respectively and let $\mathbf{N}$ be a unit normal of $T_p\Sigma_1 = T_p\Sigma_2$, L_1 and L_2 be the Weingarten maps of Σ_1 and Σ_2 with respect to $\mathbf{N}$.
- Consider an isotopy $H\colon \Sigma_1 \times (-\tau, \tau) \to M$, for which $H(q, 0) = q$ for all $q \in \Sigma_1$.



- For $t \in (-\tau, \tau)$, denote by $H_t \colon \Sigma_1 \to M$ the map $H_t \colon q \mapsto H(q,t)$, set $\Sigma_1^t = H_t(\Sigma_1)$, and let $\{D_1^t \mid |t| < \tau\}$ be the one-parameter family of regular domains bounded by Σ_1^t in M.
- Assume that the initial speed $X_p = \frac{\mathrm{d}}{\mathrm{d}\,t} H(p,t)|_{t=0}$ is not zero and points toward the interior of D_2.

Theorem

With the notation introduced above, if $\pm(L_2 - L_1)$ has positive eigenvalues, where $\pm = sgn(\langle X_p, \mathbf{N}\rangle)$, then for small positive values of t we have

$$\mu(D_1^t \cap D_2) = \frac{\omega_{n-2}}{(n^2-1)\sqrt{|\det(L_1 - L_2)|}} (2|\langle X_p, \mathbf{N}\rangle|t)^{\frac{n+1}{2}} + O\left(t^{\frac{n+2}{2}}\right).$$

Corollary

For a unit speed geodesic γ denote by $L_\gamma(r,t)$ the Weingarten map of the geodesic sphere $S(\gamma(t-r), |r|)$ at $\gamma(t)$ with respect to the unit normal $\gamma'(t)$. Then $KP_2^= \implies D(r) = \det(L_\gamma(r,t) - L_\gamma(-r,t))$ depends only on r.

Expression of $L_\gamma(r,t)$ with Jacobi fields

- Let γ be a unit speed geodesic,
- $E_1,\dots,E_{n-1},E_n=\gamma'$ be an orthonormal parallel frame along γ.
- $R(t)$ be the matrix of the Jacobi operator $R_{\gamma'(t)}: T^\perp_{\gamma(t)}M \to T^\perp_{\gamma(t)}M$, $\mathbf{v}\mapsto R(\mathbf{v},\gamma'(t))\gamma'(t)$ with respect to $E_1(t),\dots,E_{n-1}(t)$.
- For all real s, let $J(s,.)\colon \mathbb{R}\to\mathbb{R}^{(n-1)\times(n-1)}$ be the solution of the following matrix differential equation
  - $\partial_2^2 J(s,t)+R(t)J(s,t)=0$,
  - $J(s,s)=0$,
  - $\partial_2 J(s,s)=I$.

  If $\mathbf{v}\in T^\perp_{\gamma(s)}M$, and $J_{s,\mathbf{v}}$ is the Jacobi field along γ defined by the initial conditions $J_{s,\mathbf{v}}(s)=0$ and $J'_{s,\mathbf{v}}(s)=\mathbf{v}$, then $[J_{s,\mathbf{v}}(t)]=J(s,t)[\mathbf{v}]$.

Theorem

$$[L_\gamma(r,t)]=-\partial_2 J(t-r,t)J(t-r,t)^{-1}=-\left(I\frac{1}{r}+\frac{R(t)}{3}r+O(r^2)\right)$$

$$\det(L_\gamma(r,t)-L_\gamma(-r,t))=\left(-\frac{2}{r}\right)^{n-1}\left(1+\frac{\operatorname{tr}(R(t))}{3}r^2+O(r^3)\right)$$

$KP_2^= \Longrightarrow$ Einstein, analytic, D'Atri

Corollary

Every $KP_2^=$ manifold is an Einstein manifold. In particular, it is an analytic manifold with the atlas of normal coordinate charts.

Definition

A Riemannian manifold is a D'Atri space if the following equivalent conditions hold

- Geodesic symmetries are locally volume preserving.
- The volume density function $\theta_p : T_m M \hookrightarrow \mathbb{R}$ is symmetric in the origin.
- Small geodesic spheres have equal mean curvature at antipodal points.
- For any $p, q \in M$ close enough, the mean curvature of $S(p, d(p,q))$ at q is equal to the mean curvature of $S(q, d(p,q))$ at p.

Theorem

Every $KP_2^=$ space is a D'Atri space, that is $\mathrm{tr} L_\gamma(a-b, a) + \mathrm{tr} L_\gamma(b-a, b) = 0$ for any unit speed geodesic and any small values of a, b.

Proof.

Set $f(a,b) = \mathrm{tr} L_\gamma(a-b,a) + \mathrm{tr} L_\gamma(b-a,b)$
The computational proof makes intensive use of the fact that the "Wronskian"

$$J(s_1,t)^T \partial_2 J(s_2,t) - \partial_2 J(s_1,t)^T J(s_2,t)$$

is a constant matrix in t. When $t = s_1$ and $t = s_2$, we get $-J(s_2,s_1) = J(s_1,s_2)^T$.
Considering the logarithmic derivative of

$$D(r) = \det(L_\gamma(r,t) - L_\gamma(-r,t)) = \frac{\det J(t-r,t+r)}{\det J(t-r,t)\det J(t+r,t)}$$

with respect to t one obtains the identity

$$f(a,b) = f\left(a, \frac{a+b}{2}\right) + f\left(\frac{a+b}{2}, b\right).$$

Using Taylor series computation we see that $\exists C > 0$ such that for any $a', b' \in [a,b]$ we have $|f(a',b')| \le C|b'-a'|^2$. Thus

$$|f(a,b)| = \left|\sum_{i=0}^{2^k-1} f\left(a + \frac{i}{2^k}(b-a), a + \frac{i+1}{2^k}(b-a)\right)\right| \le 2^k C \left|\frac{b-a}{2^k}\right|^2.$$

□

The Final Step

Corollary

$$\mathrm{tr}(L_\gamma(-r, t-r)) = -\mathrm{tr}(L_\gamma(r, t+r))$$

Computing the logaritmic derivative of $D(r)$ with respect to r, we obtain

$$(logD)'(r) = -2\mathrm{tr}(L_\gamma(2r, t)) + \mathrm{tr}(L_\gamma(r, t))$$

Substituting the Laurent series

$$\mathrm{tr}(L_\gamma(r, t)) = -\frac{n-1}{r} + \sum_{i=1}^{\infty} a_i(t) r^i$$

we obtain

$$(logD)'(r) = -\frac{n-1}{r} + 2\sum_{i=1}^{\infty}(2^i - 1)a_i(t) r^i$$

Thus, $a_i(t)$ and $\mathrm{tr}(L_\gamma(r, t)$ does not depend on t and γ. □

Riemannian manifolds with the $KP_3^=$ property

Definition

The minimal covering radius of a subset Y of a metric space X is the infimum of those radii, for which Y can be covered by a ball of radius r.

Observation

$KP_3^=$ implies that the minimal covering radius of point triples may depend only on the distances between the points.

Theorem (B. Cs., M. Horváth)

- *If a Riemannian manifold M^n has the property that the minimal covering radius of a point triple depends only on the distances between the points, then it has constant sectional curvature.*
- *If M^n is connected and complete, then M^n is one of the spaces $\mathbb{H}^n$, $\mathbb{E}^n$, $\mathbb{S}^n$.*

Thank you for your attention!