Spindle convex inequalities

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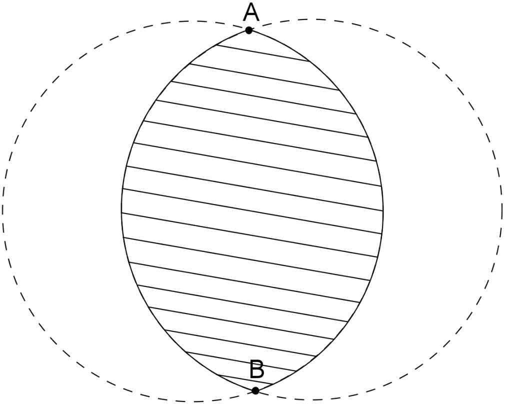
Joint work with Ferenc Fodor and Árpád Kurusa.

We work in the d-dimensional Euclidean space. The intersection of all closed unit balls containing a pair of of points A and B is called the *spindle* spanned by A and B.



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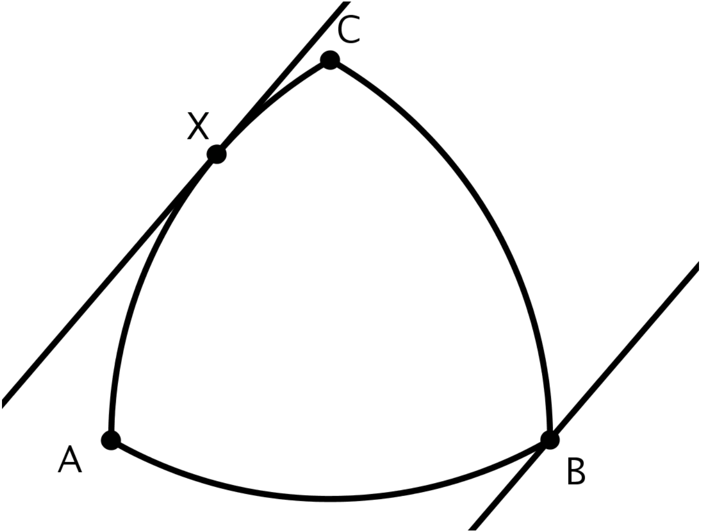
From now on S denotes a compact spindle convex set, in other words a spindle convex body. [For simplicity we consider one point as a spindle convex body.]

The Reuleaux triangle



Another example: the "Loonie"

Short history

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Recently Bezdek, Lángi, Naszódi and Papez started a more thorough study of spindle convexity (the terminology comes from their paper Ball-polyhedra), their work was continued by Kupitz, Martini and Perles.

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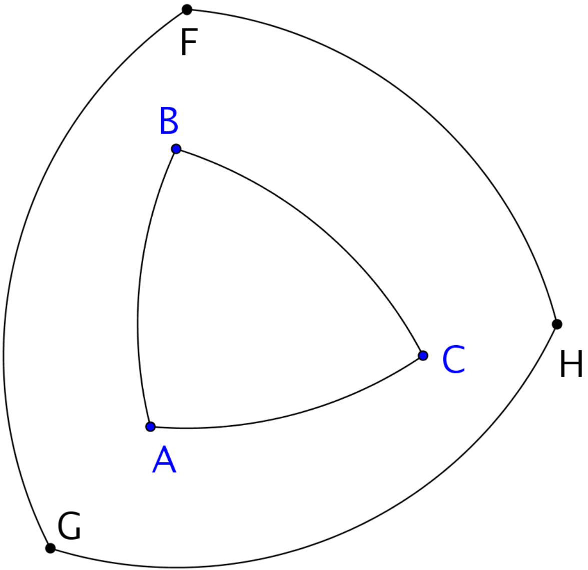
Proposition

$$\begin{aligned} S^* &= \{P \mid S \subseteq B(P)\} = \bigcap_{P \in S} B(P) \\ S^{**} &= S \\ S + (-S^*) &= B_d \end{aligned}$$

The dual of a disc-triangle



Duality and constant width

Proposition

K is a convex body of constant width 1 if, and only if, K is a selfdual spindle convex body ($K^* = K$).

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Theorem (F. Fodor, Á. Kurusa, V. V. (2013+))

$$W_k(S^*) = \sum_{j=0}^{d-k} (-1)^j \binom{d-k}{j} W_{d-j}(S)$$

As a special case we get Barbier's Theorem: a convex disc of constant width 1 has perimeter π.

Theorem [Santaló (1946)]

Amongst all spindle convex discs of a given circumradius R the regular disc-triangle has the smallest diameter.

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Theorem [Santaló (1946)]

Amongst all spindle convex discs of a given diameter D the corresponding spindle has the smallest area, while the circle has the largest area.

Blaschke-Lebesgue Theorem [1914-1915]

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Note that by the classical Isoperimetric Inequality the maximal area belongs to the circle, thus we can think of the Blaschke-Lebesgue Theorem as a reverse isoperimetric inequality.

A Reverse Isoperimetric Inequality

Theorem [F. Fodor, Á. Kurusa, V. V. (2013+)]

Fix $0 < \kappa < 2\pi$. Then amongst all spindle convex discs of perimeter κ the corresponding spindle has the smallest area.

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The statement was conjectured by K. Bezdek around 2008 in any dimension. M. Bezdek proved a similar result for fat disc-polygons in 2009. To our best knowledge there is no result in higher dimensions yet.

The Blaschke-Santaló inequlity

Tétel [Blaschke (1917, d=2,3), Santaló (1949)]

Let K be a convex body in the Euclidean d-space such that the centroid of K is at the origin. Then

$$V_d(K)V_d(K^*) \leq V(B_d)V(B_d^*) = \kappa_d^2,$$

with equality if and only if K is an ellipsoid.

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A stability version of this theorem was recently proved by Böröczky Jr. The problem of the lower bound on the volume product is known as the Mahler Conjecture, and is still not yet fully understood.

Theorem [F. Fodor, Á. Kurusa, V. V. (2013+)]

The volume product $V(S)V(S^*)$ is maximal if, and only if, S is a ball of radius $1/2$, that is S is a selfdual ball.

A Blaschke-Santaló type inequality

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Theorem [F. Fodor, Á. Kurusa, V. V. (2013+)]

For any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ with the property that if S is a spindle convex body for which

$$V(S)V(S^*) \geq \frac{(1-\delta)}{4^d}\kappa_d^2,$$

then there exists a ball B of radius $1/2$ with $\delta_H(S, B) < \varepsilon$.

We conjecture that a similar stability result holds true for the Reverse Isoperimetric Inequality stated earlier.

Lemma

For any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ with the property that if S is a **disc-triangle** of perimeter κ for which

$$A(S) \leq (1+\delta)A(\ominus_\kappa),$$

then there exists a spindle $\ominus$ of perimeter κ with $\delta_H(S, \ominus) < \varepsilon$.

NEW SZÉCHENYI PLAN

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HUNGARY'S RENEWAL

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Thank you for your attention!