THE RECENT STATUS OF THE VOLUME PRODUCT PROBLEM

Based on the paper

K. J. Böröczky, M. Meyer, S. Reisner, E. M. Jr.

Volume product in the plane — lower estimates with stability,

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Def. $K \subset \mathbb{R}^n$ convex body: compact, convex, $\operatorname{int} K \neq \emptyset$.

Not. $V(K)$: volume of K.

Not. $0 \in \operatorname{int} K$: polar body $K^* := \{x \in \mathbb{R}^n \mid \forall k \in K \ \langle x, k \rangle \leq 1\}$.

Also convex body, with $0 \in \operatorname{int} K^*$.

$(K^*)^* = K$.

K 0-symmetric $\Rightarrow K$ is unit ball of an n-dimensional Banach space X, then K^* is unit ball of dual space X' (X, X' identified via scalar product $\langle \, , \, \rangle$)

Def. Volume product: $V(K)V(K^*)$

It is invariant under non-singular linear transformations.

Originates from Blaschke, Mahler.

Turned out to be very important in the local theory of Banach spaces (asymptotic study of high finite dimensional Banach spaces), where it has relations to a number of other characteristics of these Banach spaces.

It emerges in at least 6 different mathematical disciplines.

Qu. Minimum, maximum of volume product = ?

Ex-s. 1) K = Euclidean unit ball, with volume $= \kappa_n \Rightarrow V(K)V(K^*) = \kappa_n^2$
$= n^{-n}(2e\pi + o(1))^n$.

2) K = cube $[-1,1]^n$, or regular cross-polytope $\mathrm{conv}\{\pm e_i\} \Rightarrow$
$V(K)V(K^*) = \frac{4^n}{n!} =$
$n^{-n}(4e + o(1))^n$.

Prop. $x \in \operatorname{int} K \Rightarrow V((K-x)^*) =$
$$\frac{1}{n}\int_{S^{n-1}} (h_K(u) - \langle x,u\rangle)^{-n}\,du,$$
where $h_K(u) =$ support function of $K := \max\{\langle k,u\rangle \mid k \in K\}$, for $u \in S^{n-1}$.

$\operatorname{grad}_x V((K-x)^*) =$
$$\int_{S^{n-1}} \underset{\substack{\uparrow\\ \text{vector}}}{u}\,(h_K(u) - \langle x,u\rangle)^{-n-1}\,du,$$
second differential of $V((K-x)^*)$ is a positive definite quadratic form

Cor. $\exists!\, x \in \operatorname{int} K$ $V((K-x)^*)$ is minimal

$\operatorname{dist}(x, \operatorname{bd} K) \to 0 \Rightarrow V((K-x)^*) \to \infty$

Def. This unique x is:

Santaló point of $K := s(K)$.

K 0-symmetric $\Rightarrow s(K)=0$.

In geometry it is unnatural to restrict to 0-symmetric bodies, which <u>is</u> natural in the local theory of Banach spaces.

Importance of Santaló point shows up only in asymmetric case, and also leads to proofs. unguessed in 0-symmetric case only.

Good qu. minimum, maximum of

$$V(K)V[(K-s(K))^*]=?$$

(a minimum, and a minimax problem).

This quantity is invariant under affinities.

UPPER BOUND

Th. 1. (Blaschke, Santaló, Saint Raymond, Petty, Meyer-Pajor)

$$V(K)\, V\big[(K - s(K))^*\big] \leq \kappa_n^2 ,$$

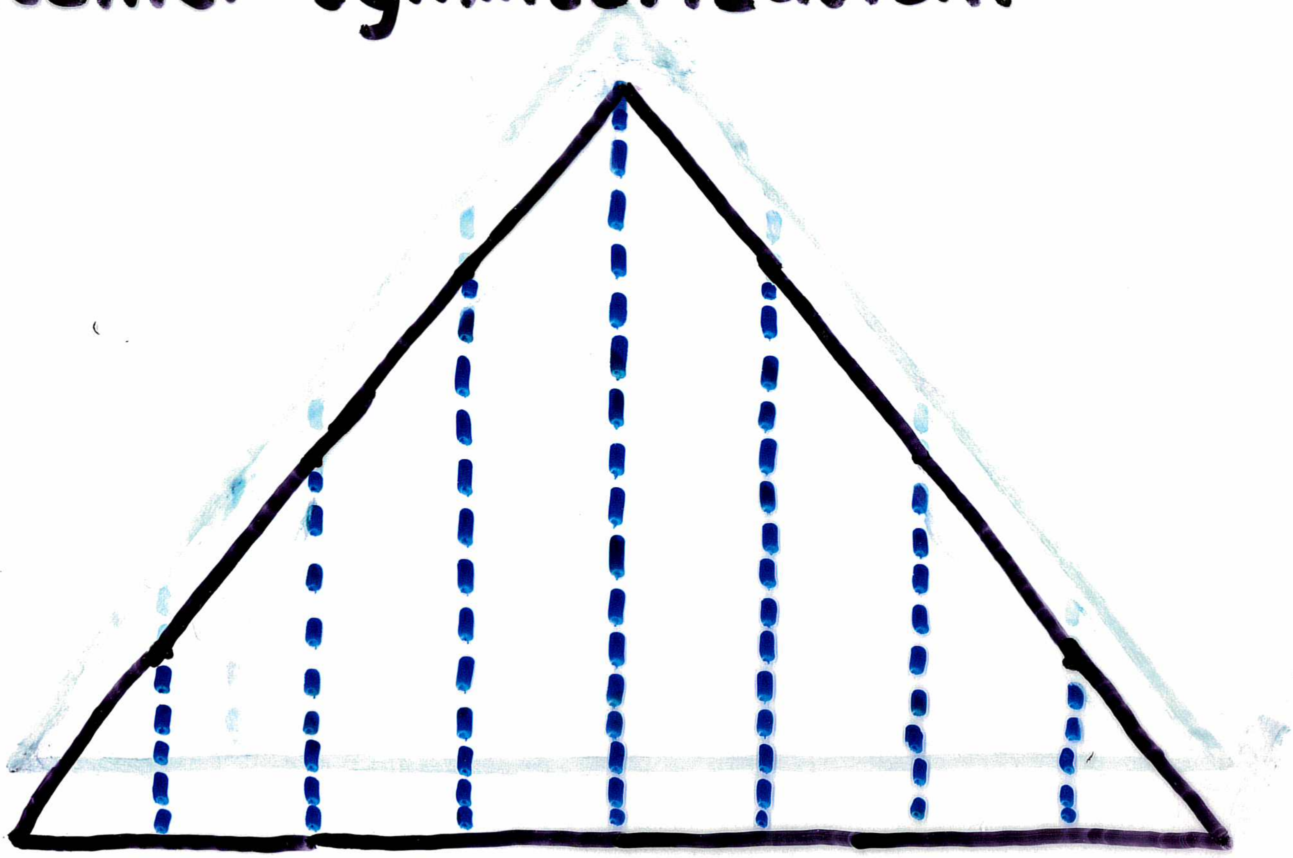
with equality exactly for an ellipsoid

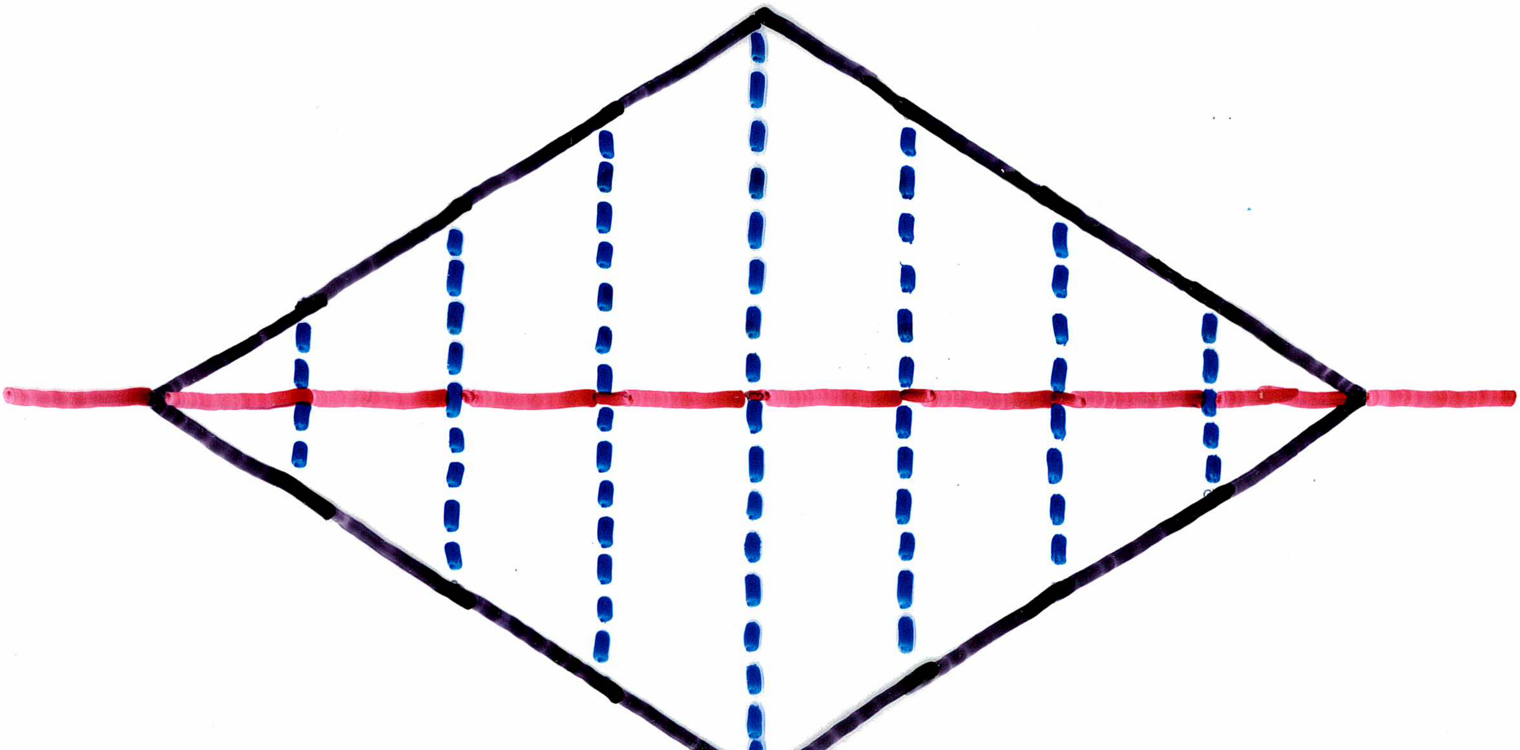
Th. 1' (Ball, Meyer-Pajor, Artstein-Avidan–Klartag–Milman, Meyer–Reisner)

Steiner symmetrization proves this. Namely, Steiner symmetrization does not decrease $V(K)V[(K-s(K))^*]$, and strictly increases it, unless K is an ellipsoid.

This is simplest proof, comparable to that of the isoperimetric inequality by Steiner symmetrization.

Steiner symmetrization:





LOWER BOUNDS

Conj. (Mahler-Guggenheimer)

K 0-symmetric $\Longrightarrow$

$$V(K)\,V\left[(K - s(K))^*\right] = V(K)\,V(K^*) \geq \frac{4^n}{n!} = n^{-n}(4e + o(1))^n,$$

with equality exactly for unit
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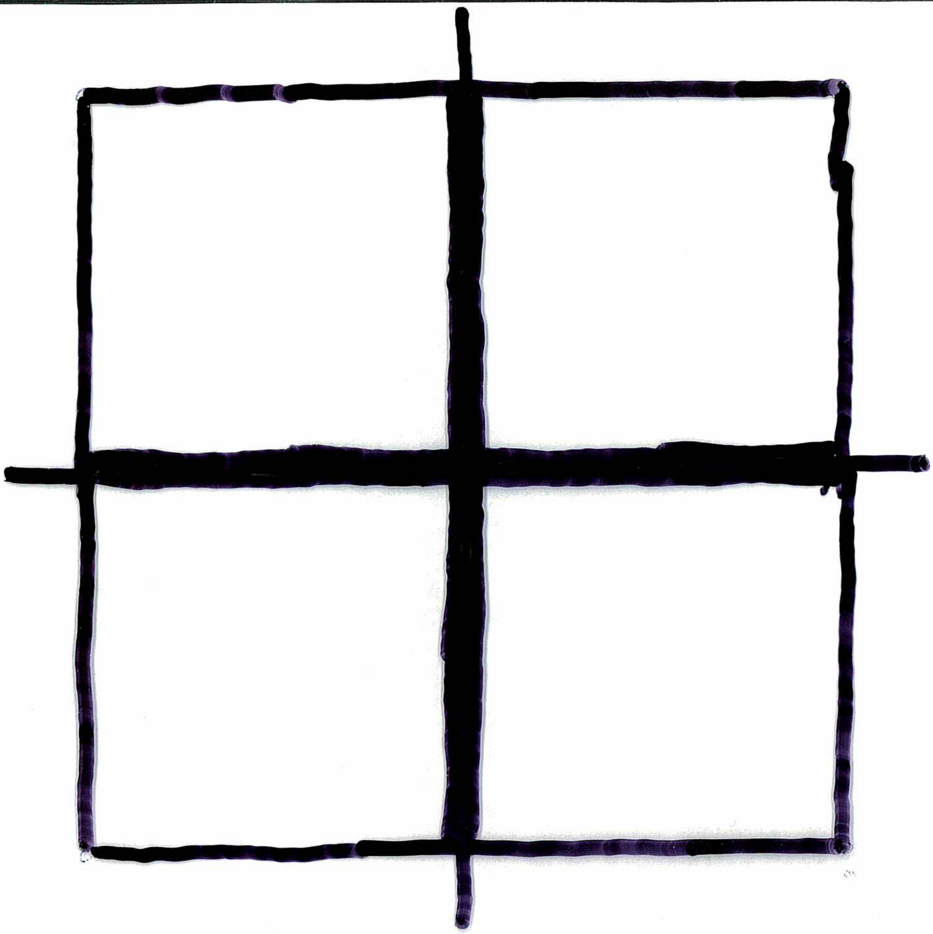
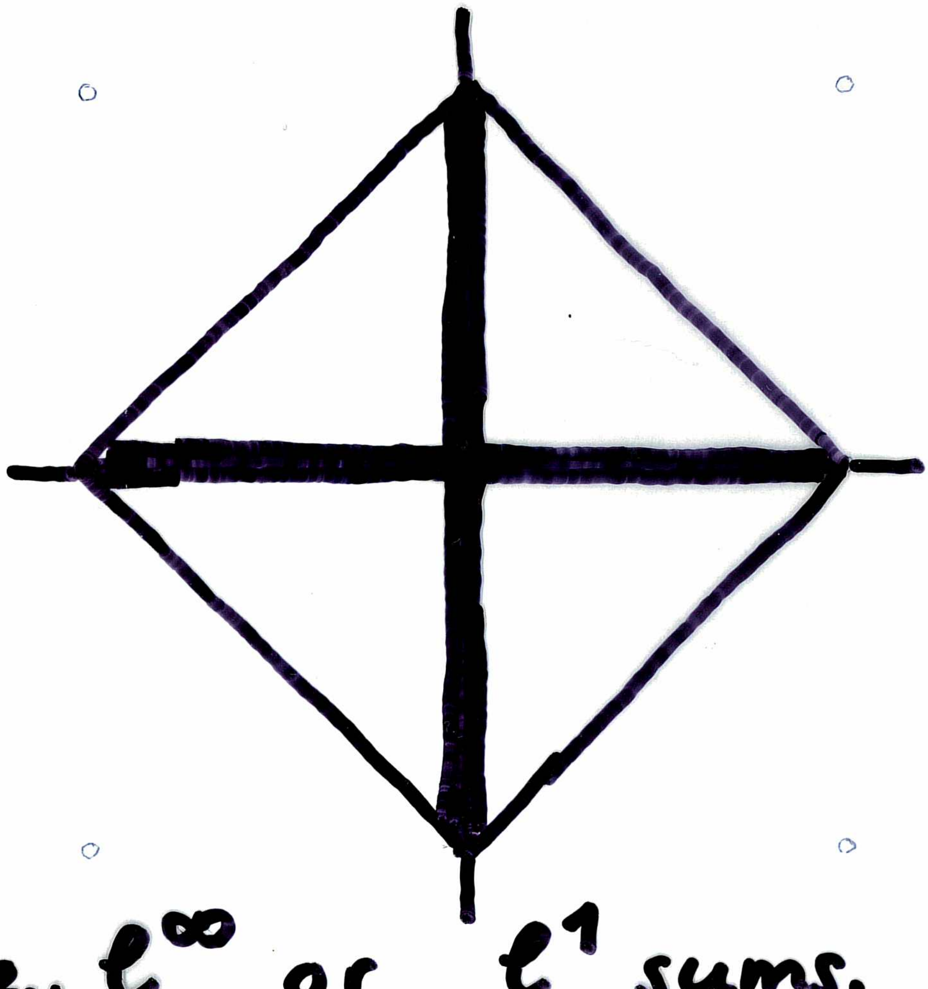
balls of Hansen-Lima Banach spaces, i.e., Banach spaces inductively defined from Banach spaces of lower dimensions, taking (Minkowski) sums, or convex hulls of unit balls.





i.e., ℓ^∞ or ℓ^1 sums,
beginning with $n=1$.

Conj. (Mahler)

$$V(K)\,V\left[\left(K-s(K)\right)^*\right] \geq \frac{(n+1)^{n+1}}{(n!)^2} = n^{-n}\left(e^2+o(1)\right)^n,$$

with equality exactly for a simplex.

General Lower Bounds

Th. 2A. (Bourgain-Milman, G. Kuperberg)

K 0-symmetric $\Rightarrow$

$$V(K)V(K^*) \geq \frac{\kappa_n^2}{2^n} = n^{-n}(e\pi + o(1))^n$$

Th. 2B. (Bourgain-Milman, G. Kuperberg)

$$V(K)V[(K-s(K))^*] \geq \text{const} \cdot n^{-n}(\pi e/2)^n$$

Sharp Lower Bounds For Special Bodies

Bodies with high symmetry

Th. 3A. (Saint Raymond, Meyer, Reisner)

K is symmetric w.r.t. all coordinate hyperplanes (thus is 0-symmetric), also called **unconditional body** (corresponding to an **unconditional norm**) $\Rightarrow$

$$V(K)V(K^*) \geq \frac{4^n}{n!},$$

with equality exactly for the conjectured cases, i.e., Hansen-Lima spaces.

Th. 3B (Barthe-Fradelizi)

K has all symmetries of a regular simplex $\Rightarrow$

$$V(K)\,V[(K-s(K))^*] \geq \frac{(n+1)^{n+1}}{(n!)^2},$$

with equality e.g. for a simplex.

Zonoids

Th. 4. (Reisner)

K is 0-symmetric, zonoid, i.e., a limit in the Hausdorff metric of finite sums of segments $\Rightarrow V(K)V(K^*) \geq \frac{4^n}{n!}$, with equality exactly for a parallelotope.

(The other conjectured equality cases are not zonoids.)

Cor. (Mahler-Reisner)

$n=2$, K is 0-symmetric $\Rightarrow$ $V(K)V(K^*) \geq 8$, with equality exactly for a parallelogram.

Planar case

Th.5. (Mahler-Meyer)

$$n=2 \Rightarrow V(K)\,V\big[(K-s(K))^*\big] \geq \frac{27}{4},$$

with equality exactly for a triangle.

Local minima

Th. 6 A. (Nazarov–Petrov–Rjabogin–Zvavič)

Among 0-symmetric K's, parallelotope has a strictly locally minimal volume product.

Th. 6 B. (Kim–Reisner)

Simplex has a strictly locally minimal volume product.

Polyhedra with a small number of vertices or $(n-1)$-faces

Th. 7A. (Lopez-Reisner)

$n \leq 8$, K is 0-symmetric polytope, with at most $n+1$ opposite pairs of vertices, or $(n-1)$-faces $\Rightarrow V(K)V(K^*) \geq \frac{4^n}{n!}$, with equality exactly for conjectured equality cases, i.e., Hansen-Lima spaces.

Th. 7B. (Meyer-Reisner)

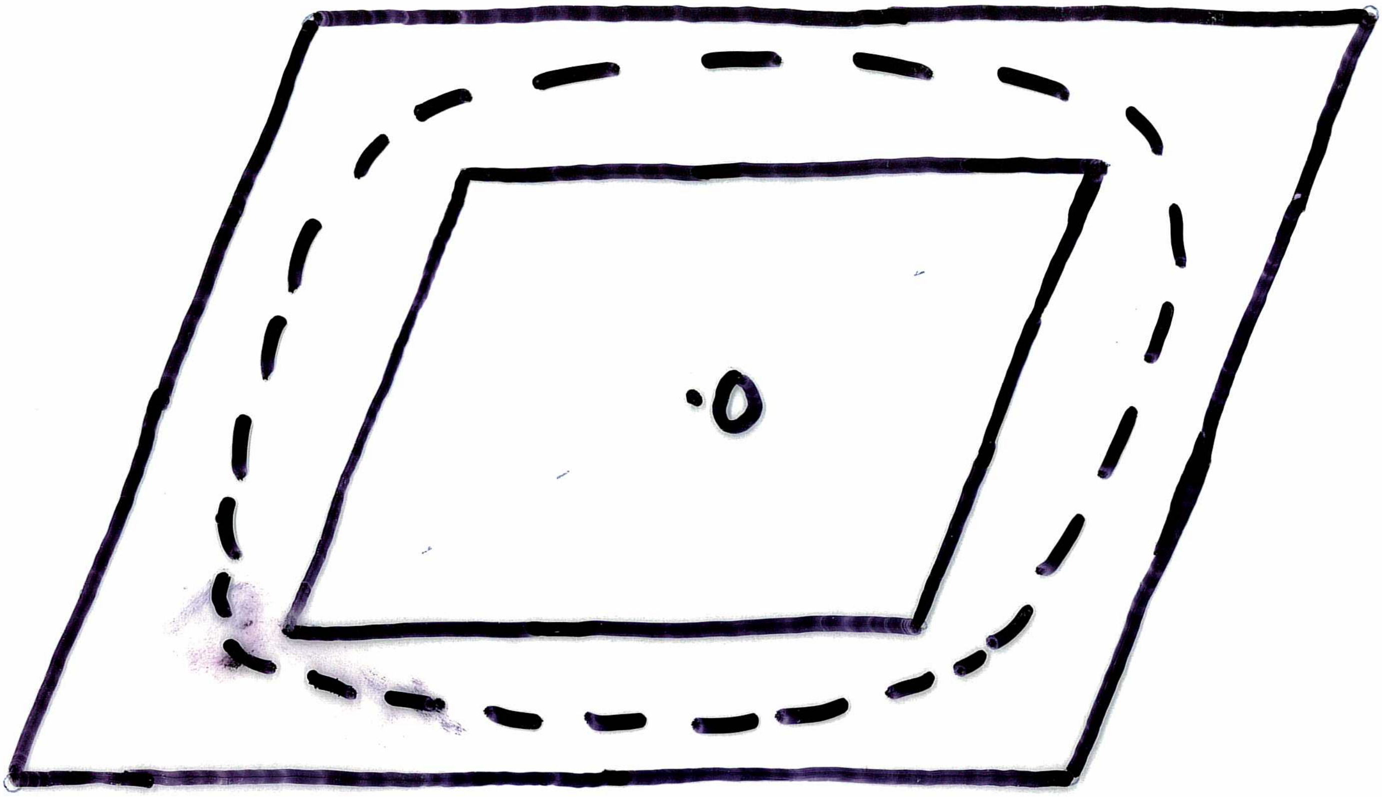
K is a polytope, with at most $n+3$ vertices or $(n-1)$-faces $\Rightarrow$

$$V(K)V[(K-s(K))^*] \geq \frac{(n+1)^{n+1}}{(n!)^2},$$

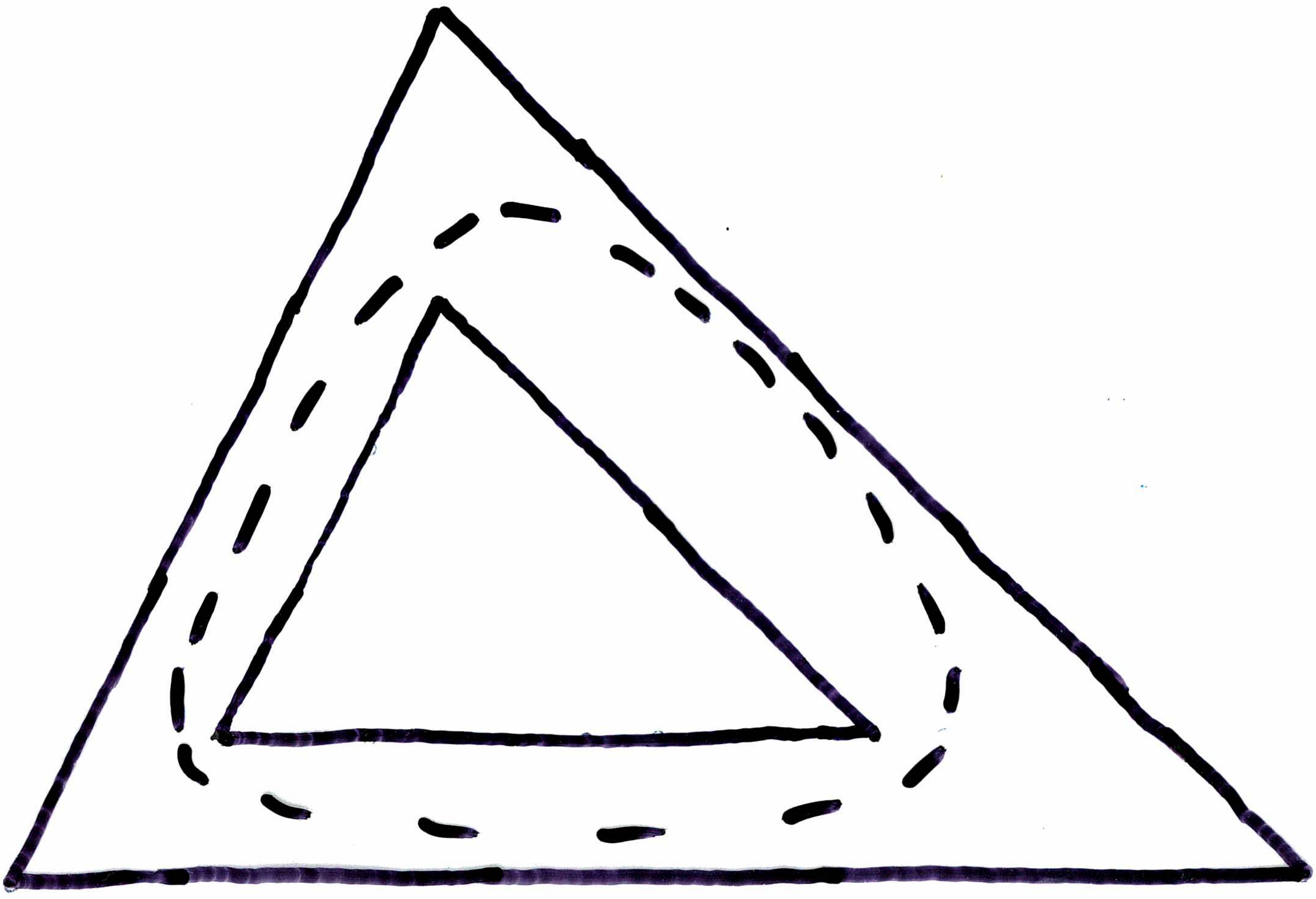
with equality exactly for a simplex.

STABILITY VARIANTS

Several of the above theorems do not only determine the equality cases, but also, if we have a body with volume product ε-close to the extremal value, then the body is $f(\varepsilon)$-close to an extremal body, where $f(\varepsilon)$ is typically a power of ε.



0-symmetric case



general case

Def. $K, L \subset \mathbb{R}^n$ convex bodies:

$\delta_{BM}(K,L) =$

Banach-Mazur distance

of K and L :=

$\min\{\lambda_2/\lambda_1 \mid \lambda_i > 0,$

$\exists A$ affinity, $\exists x, y \in \mathbb{R}^n$,

$\lambda_1 K + x \subset AL \subset \lambda_2 K + y$

For K, L 0-symmetric this is Banach-Mazur distance of Banach spaces with unit balls K, L.

We have stability of the upper bound result, for $n \geq 3$

- where probably the order of $f(\varepsilon)$ is not optimal.

We have stability, among the lower bound results, for zonoids, for the planar case, and for the local minima

- for each of which the order of $f(\varepsilon)$ is optimal.

FUNCTIONAL VARIANTS

One can consider, rather than convex bodies, log-concave functions $f: \mathbb{R}^n \to \mathbb{R}$, i.e., $f \geq 0$, and $\log f$ is concave.

A convex body K, with $0 \in \operatorname{int} K$ is mapped to $f(x) := e^{-\|x\|_K^2/2}$ (where $\|\cdot\|_K$ is the asymmetric norm with unit ball K).

Then volume goes over to

$\text{const}_n \cdot \int_{\mathbb{R}^n} f(x)\,dx.$

Polarity $K \leftrightarrow K^*$ goes over to $-\log f$, $-\log f^*$ being Legendre transforms of each other.

Def. **Legendre transform** of $\varphi: \mathbb{R}^n \to \mathbb{R}$:

$\mathcal{L}\varphi$, defined by

$$(\mathcal{L}\varphi)(x) := \sup\{\langle x, y\rangle - \varphi(y) \mid y \in \mathbb{R}^n\}.$$

Unfortunately, to translations of K (preserving $0 \in \operatorname{int} K$) there are no corresponding good transformations. So upper estimates follow only in the 0-symmetric case (f even), when the function corresponding to the Euclidean unit ball gives the maximum (thus implies Blaschke-Santaló inequality, 0-symmetric case).

For lower estimate no translations are needed (that is question of $\min\{V(K)V(K^*) \mid 0 \in \operatorname{int} K\}$).

Then the functional inequality corresponding to the inequality $V(K)V(K^*) \geq n^{-n} \cdot \mathrm{const}^n$ holds (thus implies inverse Blaschke-Santaló inequality).