Generalized Minkowski space

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In memory of Hermann Minkowski

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$\boxed{\text{Euclidean space}}$

$\Uparrow$

$\boxed{\text{inner-product}}$

$\nearrow \qquad \nwarrow$

$\boxed{\text{semi-inner-product}} \qquad \boxed{\text{indefinite-inner-product}}$

$\Downarrow \qquad \Downarrow$

$\boxed{\text{normed space}}$ and $\boxed{\text{pseudo-euclidean space}}$

$\searrow \qquad \swarrow$

$\boxed{\text{Minkowski space}}$

The $\left\{ \begin{array}{r} semi-inner-product \\ indefinite-inner-product \end{array} \right.$ on a complex vector space V is a complex function $[x, y] : V \times V \longrightarrow \mathbb{C}$ with the properties $\left\{ \begin{array}{r} s1, s2, s3, s4 \\ i1, i2, i3, i4 \end{array} \right.$

s1=i1: $[x + y, z] = [x, z] + [y, z]$ (additivity of the first argument)

s2=i2: $[\lambda x, y] = \lambda [x, y]$ for every $\lambda \in \mathbb{C}$ (homogeneity of the first argument)

s3: $[x, x] > 0$ when $x \neq 0$ (positivity)

i3: $[x, y] = \overline{[y, x]}$ for every $x, y \in V$ (antisymmetry)

s4: $|[x, y]|^2 \leq [x, x][y, y]$ (Cauchy-Schwartz inequality)

i4: $[x, y] = 0$ for every $y \in V$ then $x = 0$. (nondegeneracy)

History of these concepts

Semi-inner-product

raised by G.Lumer in 1961, in 1967 J.R. Giles prove that the property

s5: $[x, \lambda y] = \bar{\lambda}[x, y]$ for all complex λ (homogeneity in the second argument)

can be imposed.

Indefinite-inner-product

first used by Minkowski, Lorentz, Einstein at the beginning of the twentieth century in the theoretical physics, the first application in mathematics (to the theory of zones of stability for canonical differential equations with periodic coefficients) were obtained by M.G.Krein in 1964,

I.M.Gelfand, N.Levinson, I.Gohberg,...

Semi-indefinite inner product

Definition 1 *The semi-indefinite inner product (s.i.i.p.) on a complex vector space V is a complex function $[x, y] : V \times V \longrightarrow \mathbb{C}$ with the following properties:*

1 $[x + y, z] = [x, z] + [y, z]$ *(additivity in the first variable),*

2 $[\lambda x, y] = \lambda[x, y]$ *for every* $\lambda \in \mathbb{C}$ *(homogeneity in the first variable),*

3 $[x, \lambda y] = \overline{\lambda}[x, y]$ *for every* $\lambda \in \mathbb{C}$ *(homogeneity in the second variable),*

4 $[x, x] \in \mathbb{R}$ *for every* $x \in V$ *(the corresponding quadratic form is real-valued),*

5 *if either* $[x, y] = 0$ *for every* $y \in V$ *or* $[y, x] = 0$ *for all* $y \in V$, *then* $x = 0$ *(nondegeneracy),*

6 $|[x, y]|^2 \leq [x, x][y, y]$ *holds on non-positive and non-negative subspaces of V, respectively. (the Cauchy-Schwartz inequality is valid on positive and negative subspaces, respectively).*

A vector space V with a s.i.i.p. is called an s.i.i.p. space.

Examples

- Normed spaces, pseudo-Euclidean spaces
- General construction for semi-indefinite inner product spa

C is the unit sphere of a normed linear space, $P(C) = C/\sim$.
By the Hahn-Banach theorem there exists at least one continuous linear functional, and we choose exactly one such that $\|\widetilde{v}^\star\| = 1$ and $\widetilde{v}^\star(v) = 1$ for $v \in C$.
Consider a sign function $\varepsilon(v)$ with value ± 1 on $P(C)$,
and if $\varepsilon([v]) = 1$ denote by $v^\star = \widetilde{v}^\star$
and if $\varepsilon([v]) = -1$ define $v^\star = -\widetilde{v}^\star$,
homogeneously extend it to V the mapping $v \mapsto v^\star$ by the equality $(\lambda v)^\star = \overline{\lambda} v^\star$.
For the duality mapping
$v \mapsto v^\star$ the equalities $v^\star(v) := \varepsilon([v_0])\|v\|^2$ and $\|v\| = \|v^\star\|$ are hold.
$\boxed{[u, v] = v^\star(u)}$ satisfies **1**-**5**.

$$|[u, v]| = |v^\star(u)| = \frac{|v^\star(u)|}{\|u\|}\|u\| \leq \|v^\star\|\|u\| = \|v\|\|u\|$$

Minkowski product

Definition 2 ([7]) *Let* $(V, [\cdot,\cdot])$ *be an s.i.i.p. space. Let* $S, T \leq V$ *be positive and negative subspaces, where* T *is a direct complement of* S *with respect to* V. *Define a product on* V *by the equality* $[u, v]^+ = [s_1 + t_1, s_2 + t_2]^+ = [s_1, s_2] + [t_1, t_2]$, *where* $s_i \in S$ *and* $t_i \in T$, *respectively. Then we say that the pair* $(V, [\cdot,\cdot]^+)$ *is a* generalized Minkowski space *with* Minkowski product $[\cdot,\cdot]^+$. *We also say that* V *is a real generalized Minkowski space if it is a real vector space and the s.i.i.p. is a real valued function.*

Example

Generalized Minkowski spaces generated by L_p norms

$$[u, v]^+ := [s_1, s_2]_S - [t_1, t_2]_T$$

$$[s_1, s_2]_S = \frac{1}{\|s_2\|_{p_1}^{p_1-2}} \int_X s_1 |s_2|^{p_1-1} \mathrm{sgn}\,(s_2) \mathrm{d}\mu$$

$$[t_1, t_2]_T = \frac{1}{\|t_2\|_{p_2}^{p_2-2}} \int_Y t_1 |t_2|^{p_2-1} \mathrm{sgn}\,(t_2) \mathrm{d}\nu$$

$$[u, v]^- := [s_1, s_2]_S + [t_1, t_2]_T$$

Generalized space-time model

Definition 3 ([7]) *Let V be a generalized Minkowski space. Then we call a vector* space-like, light-like, *or* time-like *if its scalar square is positive, zero, or negative, respectively. Let $\mathcal{S}, \mathcal{L}$ and $\mathcal{T}$ denote the sets of the space-like, light-like, and time-like vectors, respectively.*

In a finite dimensional, real generalized Minkowski space with $\dim T = 1$ we can geometrically characterize these sets of vectors. Such a space is called *generalized space-time model.* In this case $\mathcal{T}$ is a union of its two parts, namely

$$\mathcal{T} = \mathcal{T}^+ \cup \mathcal{T}^-,$$

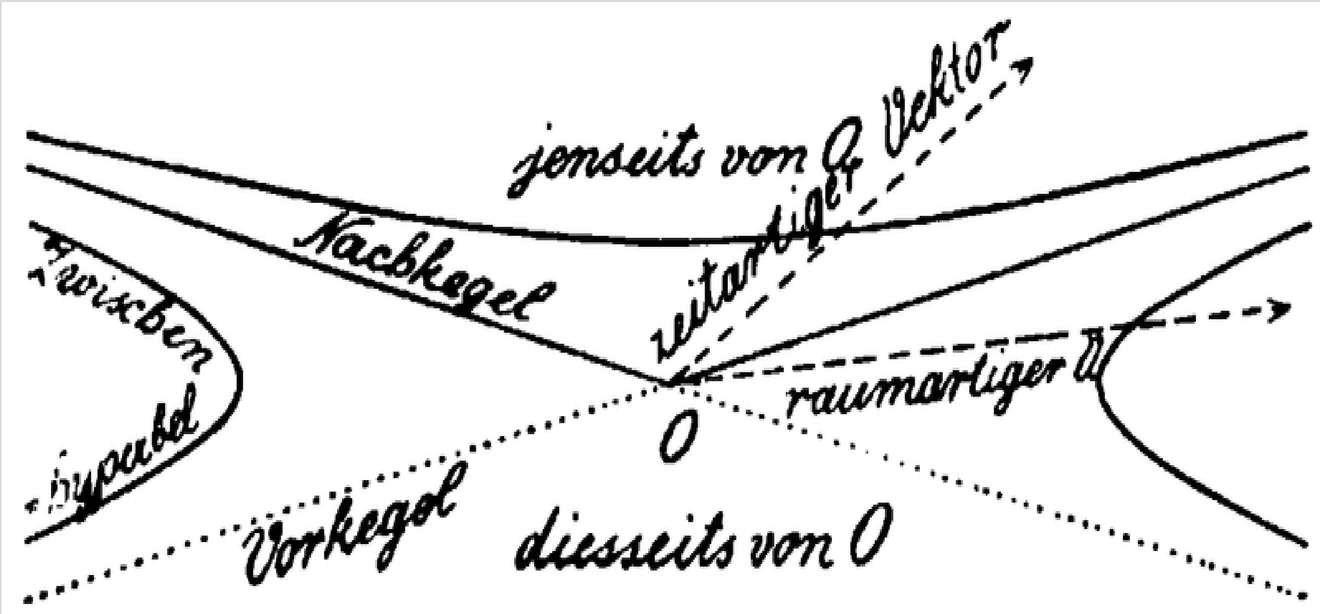
where with respect a basis with time-like vector $e_n \in T$

$$\mathcal{T}^+ = \{s + t \in \mathcal{T} | \text{ where } t = \lambda e_n \text{ for } \lambda \geq 0\} \text{ and}$$

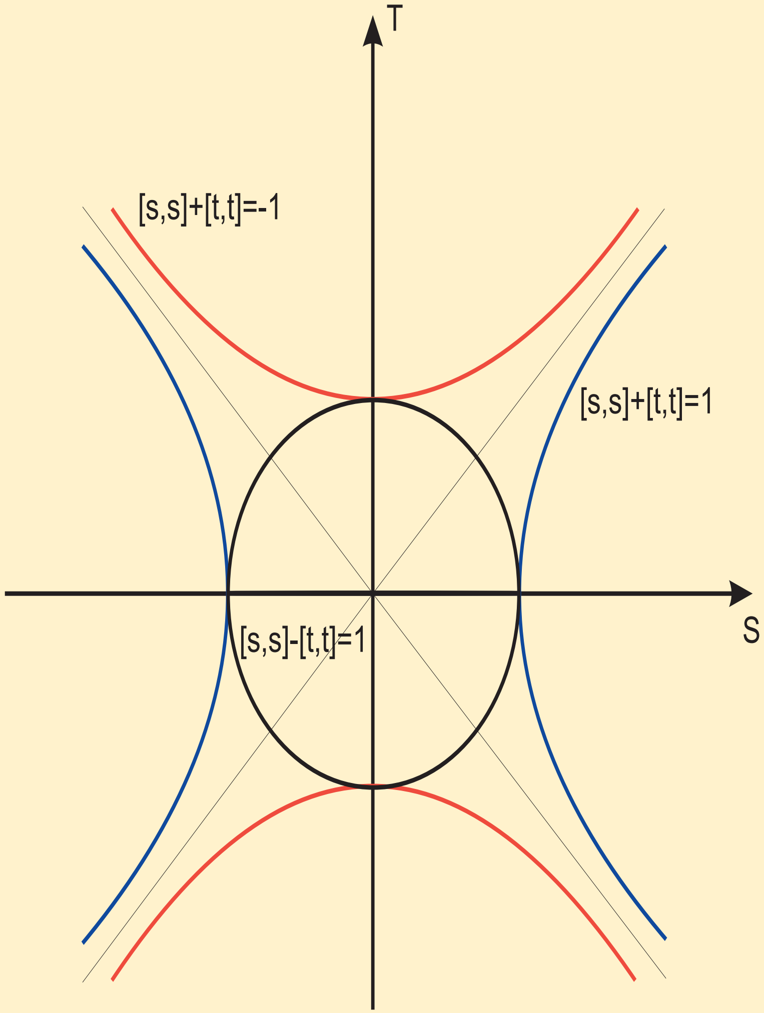
$$\mathcal{T}^- = \{s + t \in \mathcal{T} | \text{ where } t = \lambda e_n \text{ for } \lambda \leq 0\}.$$

It can be proved that $\mathcal{T}$ is an open double cone with boundary $\mathcal{L}$, and the positive part $\mathcal{T}^+$ (resp. negative part $\mathcal{T}^-$) of $\mathcal{T}$ is convex.

The Minkowski space-time



Unit spheres in space-time



Prehyperbolic space in generalized space-time

The set $H := \{v \in V | [v, v]^+ = -1\}$ is called the *imaginary unit sphere* of the generalized space-time model. H^+ is the connected part of H defined by the function

$$\mathfrak{h} : s \longmapsto \sqrt{1 + [s, s]}.$$

The geometric properties of H^+, using the differential geometry of a generalized space-time model, can be listed as follows:

- Let S be a continuously differentiable s.i.p. space, then (H^+, ds^2) is a Minkowski-Finsler space (see this concept in [7]).
- H^+ is always convex. It is strictly convex if and only if the s.i.p. space S is a strictly convex space.([8])
- If S is a continuously differentiable s.i.p. space then H^+ has constant negative curvature.([8])

We can regard H^+ as a natural generalization of the usual hyperbolic space. Thus we can say that H is a premanifold with constant negative curvature and

Presphere

$$g(s) = s + \mathfrak{g}(s)e_n,$$

where

$$\mathfrak{g}(s) = \sqrt{-1 + [s, s]} \text{ for } [s, s] > 1.$$

The results on G^+ are the following:

- G^+ and its tangent hyperplanes are intersecting, consequently there is no point at which G would be convex.([8])
- The de Sitter sphere G has constant positive curvature if S is a continuously differentiable s.i.p space.([8])

On the basis of this theorem we can tell about G as a premanifold of constant positive curvature and we may say that it is a *presphere*.

Light cone

$$l(s) = s + \sqrt{[s, s]}e_n.$$

- The light cone L^+ has zero curvatures if S is a continuously differentiable s.i.p space.([8])

Hence L is a premanifold with zero sectional, Ricci and scalar curvatures, respectively. We may also say that it is a *pre-Euclidean* space.

The unit sphere of the embedding s.i.p. space

$$k(s) = s + \mathfrak{t}(s)e_n,$$

where

$$\mathfrak{t}(s) = \sqrt{1 - [s,s]} \text{ for } [s,s] < 1.$$

The basic properties of K^+ are

- K^+ is convex. If S is a strictly convex space, then K^+ is also strictly convex.
- The fundamental forms are

$$\mathrm{I} = [\dot{c},\dot{c}] - \frac{\left([\dot{c}(t), c(t)] + [c(t),\cdot]'_{\dot{c}(t)}(c(t))\right)^2}{4(1 - [c(t),c(t)])} = [\dot{c},\dot{c}] - \frac{[\dot{c}(t),c(t)]^2}{1-[c(t),c(t)]},$$

$$\mathrm{II} = \frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}\left(-[\dot{c}(t),\dot{c}(t)] + \frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}\right) =$$

$$= -\frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}\mathrm{I}.$$

- The principal, sectional, Ricci and scalar curvatures at a point $k(c(t))$ are

$$\rho_{\max}(u,v) = \rho_{\min}(u,v) = -\frac{1}{\sqrt{|-1+2[c(t),c(t)]|}},$$

$$\kappa(u,v) := [n^0(c(t)), n^0(c(t))]^+ \rho(u,v)_{\max}\rho(u,v)_{\min} = \frac{1}{-1+2[c(t),c(t)]},$$

$$\mathrm{Ric}(v)_{k(c(t))} := (n-2)\cdot E(\kappa_{k(c(t))}(u,v)) = \frac{n-2}{-1+2[c(t),c(t)]},$$

and

$$\Gamma_{k(c(t))} := \binom{n-1}{2}\cdot E(\kappa_{f(c(t))}(u,v)) = \frac{\binom{n-1}{2}}{-1+2[c(t),c(t)]},$$

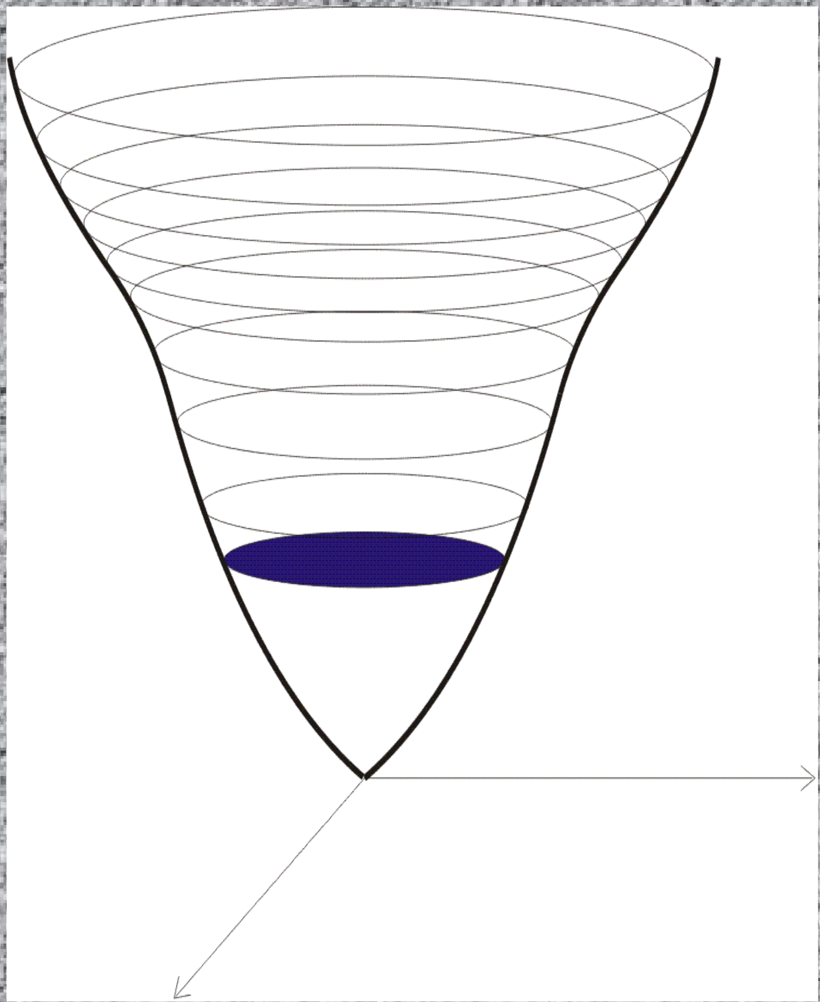
respectively.

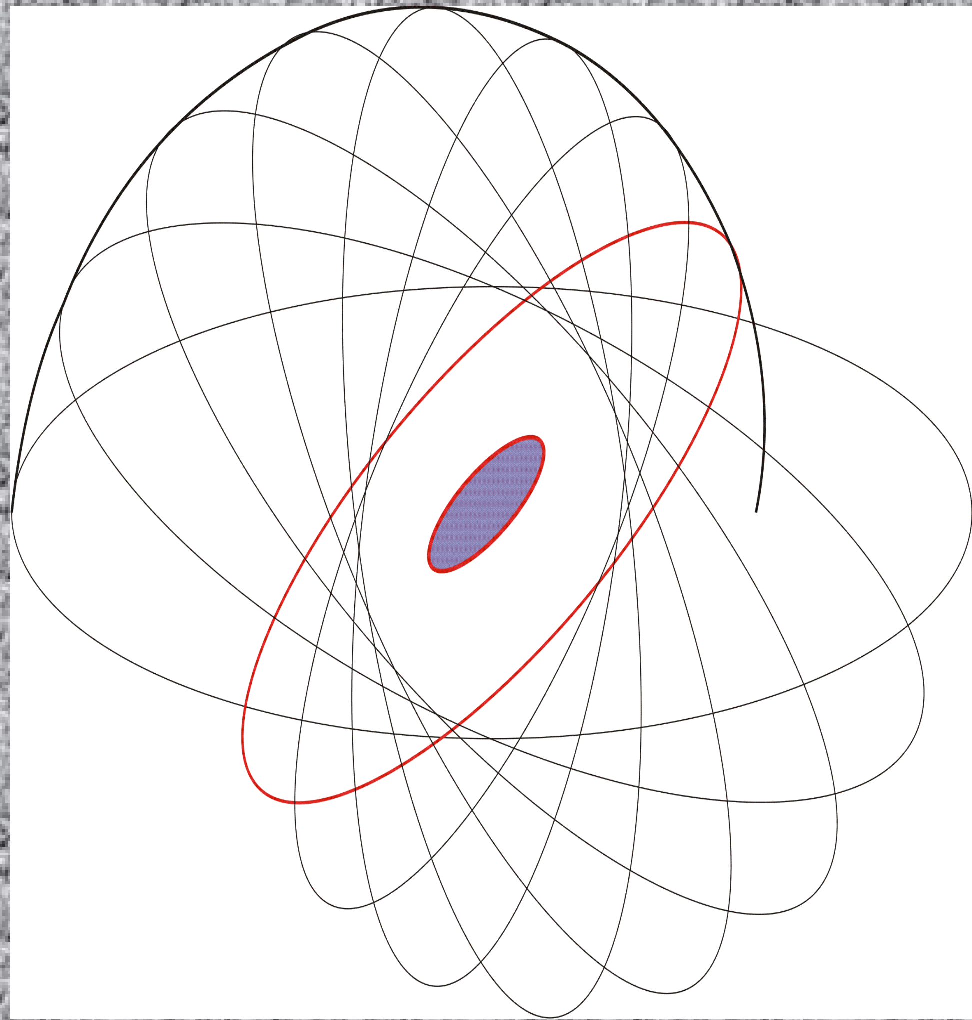
- Finally we remark that at the points of K^+ having the equality

$$2[c(t),c(t)] = 1$$

all of the curvatures can be defined as in the case of the light cone and can be regarded as zero.

Changing shape





Deterministic time-space model

$\tau K(\tau) \subset \{S, \|\cdot\|^{\tau}\}$.

These are

- $K(\tau)$ is a centrally symmetric, convex, compact, C^2 body of volume $\mathrm{vol}(B_E)$.
- For each pairs of points s', s'' the function

$$K : \mathbb{R}^+ \cup \{0\} \to \mathcal{K}_0 \ , \tau \mapsto K(\tau)$$

  holds the property that $[s', s'']^{\tau} : \tau \mapsto [s', s'']^{\tau}$ is a C^1-function.

Definition 10 *We say that a generalized space-time model endowed with a function $K(\tau)$ holding the above properties is a* deterministic time-space model.

Random time-space model

Definition 11 *Let* $(K_\tau \ , \tau \geq 0)$ *be a random function defined as an element of the Kolmogorov's extension* $\left(\Pi\mathcal{K}_0, \hat{P}\right)$ *of the probability space* $(\mathcal{K}_0, P)$*. We say that the generalized space-time model with the random function*

$$\hat{K}_\tau := \sqrt[n]{\frac{\mathrm{vol}(B_E)}{\mathrm{vol}(K_\tau)}}K_\tau$$

is a random time-space model*. Here* $\alpha_0(K_\tau)$ *is a random variable with truncated normal distribution and thus* $(\alpha_0(K_\tau) \ , \tau \geq 0)$ *is a stationary Gaussian process. We call it the* shape process *of the random time-space model.*

The probability measure P (I)

$$\delta^h(C,D) = \max\left\{\max_{x\in C}\min_{y\in D}\|x-y\|, \max_{y\in D}\min_{x\in C}\|x-y\|\right\} \text{ for } C, D \in \mathcal{K}.$$

$$\mathcal{K}_0^1 := \{K \in \mathcal{K}_0 \mid \delta^h(K, B_E) = 1\}$$

To this (following Hoffmann's paper) we introduced the orbits of a body K about the special orthogonal group $SO(n)$ by $[K]$. These are compact subsets of $\mathcal{K}_0^1$, and if we consider an open subset of $\mathcal{K}_0^1$ then the union of the corresponding orbits is also open. Hence there exists a measurable mapping $s : \mathcal{K}_0^1 \longrightarrow \mathcal{K}_0^1$ such that $s(K) = s(K')$ if and only if K and K' are on the same orbit. Let $\widetilde{\mathcal{K}_0^1} := \{K \in \mathcal{K}_0^1 \ , \ s(K) = K\}$ which is measurable subset of $\mathcal{K}_0^1$. We equip it with the induced topology of $\mathcal{K}_0^1$. Finally let $\Phi_{2a}^1 : \widetilde{\mathcal{K}_0^1} \times SO(n) \longrightarrow \mathcal{K}_0^1$ is the mapping defined by the equality:

$$\Phi_{2a}^1(K, \Theta) = \Theta K.$$

[46] Hoffmann, L. M.: Measures on the space of convex bodies. *Adv. Geom.* **10** (2010), 477–486.

Our notation is analogous with the notation of [46]. It was proved in [46] (Lemma 2) that a non-trivial σ-finite measure μ_0 on $\mathcal{K}_0$ is invariant under rotations (meaning that for $\Theta \in SO(n)$ we have $\mu_0(\mathcal{A}) = \mu(\Theta\mathcal{A})$ for all Borel sets $\mathcal{A}$ of $\mathcal{K}_0$) if and only if there exists a σ-finite measure $\widetilde{\mu_0}$ on $\widetilde{\mathcal{K}_0}$ such that $\mu_0 = \Phi_{2a}(\widetilde{\mu_0} \otimes \nu_n)$, where ν_n is the Haar measure on $SO(n)$. It is obvious that in the case of $\mathcal{K}_0^1$ there is a similar result by our mapping $\Phi_{2a}^1(K, \Theta)$ which is the restriction of Hoffmann's map $\Phi_{2a}(K, \Theta)$ onto the set $\mathcal{K}_0^1$.

The probability measure P (II)

First we chose a countable system of bodies K_m to define a probability measure on $\widetilde{\mathcal{K}_0^1}$. Without loss of generality we may assume that each of the bodies of $\widetilde{\mathcal{K}_0^1}$ has a common diameter of length 4 denoted by d, which lies on the n^{th} axe of coordinates (hence it is the convex hull of the points $\{2e_n, -2e_n\}$). Consider the set of diadic rational numbers in $(0, 2]$. We can write them as follows:

$$\left\{ m(n,k) := \frac{k}{2^n} \text{ where } n = 0, \cdots \infty \text{ and for a fixed } n,\ 0 < k \leq 2^{n+1} \right\}.$$

Define the body $K_{m(n,k)}$ as the convex hull of the union of the segment d and the ball around the origin with radius $m(n,k)$. For each n we have 2^{n+1} such bodies, thus the definition

$$\widetilde{\mu_0^1} := \lim_{n\to\infty} \sum_{k=1}^{2^{n+1}} \frac{1}{2^{n+1}} \delta_{K_{m(n,k)}}$$

define a probability measure on $\widetilde{\mathcal{K}_0^1}$.

Lemma 3.8 ([9]). *The pushforward measure $w(K)^{-1}(\widetilde{\mu_0^1})$ has uniform distribution on the interval $(0, 4]$. ($w(K)$ means the width of the body K.)*

The probability measure P (III)

$$\alpha_0(K) = \frac{d(K)}{w(K) + d(K)}.$$

Lemma 3.7 ([9]). *If $K \in \mathcal{K}_0^1$ and $\alpha_0 := \alpha_0(K)$ is the thinness of K then we have*

$$\delta^h(\alpha K, B_E) = \begin{cases} 2\alpha - 1 & \text{if} \quad \alpha_0 \le \alpha \\ 2\alpha + 1 - 2\frac{\alpha}{\alpha_0} & \text{if} \quad 0 \le \alpha < \alpha_0. \end{cases}$$

$$G(X)\mathrm{d}\lambda^{\mathrm{n}^2} := \frac{1}{\left(\sqrt{2\pi}\right)^{n^2}} e^{-\frac{1}{2}\mathrm{Tr}(X^T X)} \mathrm{d}\lambda^{\mathrm{n}^2}.$$

The Haar measure ν_n of $\mathbb{R}^{n\times n}$ is the pushforward

measure of the Gaussian measure by the mapping M defined by the Gram-Schmidt process (see in [55]).

Theorem 3.10 ([9]). *Let define the measure $\widetilde{\nu_0^1}$ by density function $\mathrm{d}\widetilde{\nu_0^1} = \frac{4}{(w+4)^2}\mathrm{d}\widetilde{\mu_0^1}$. Then*

$$\alpha_0(K)^{-1}\left(\Phi_{2a}^1\left(\widetilde{\nu_0^1} \otimes \nu_n\right)\right)$$

is a probability measure with uniform distribution on $[\frac{1}{2}, 1)$.

The probability measure P (IV)

Let denote by ν_0^1 the measure $\Phi_{2a}^1\left(\widetilde{\nu_0^1}\otimes\nu_n\right)$

We identified $\mathcal{K}_0$ with $\mathcal{K}_0^1\times[0,\infty)$, and introduced Φ_4 as the mapping $\Phi_4:(K,\alpha)\mapsto\alpha K$.

Lemma 3.9. *From the image $K'=\Phi_4(K)$ we can determine uniquely the body K and the constant α.*

$$\Phi_4^{-1}(K'):=\left(\left(\Phi_4^{-1}\right)_1(K'),\left(\Phi_4^{-1}\right)_2(K')\right)$$

$\nu_0=\Phi_4(\nu_0^1\otimes\nu)$. where ν is a σ-finite measure on $(0,\infty)$.

$$p(\mathcal{A}):=\frac{1}{\sqrt{2\pi\sigma^2}}\int\limits_{K'\in\mathcal{A}}e^{-\frac{\left(\delta^h\left(B_E,\frac{\alpha_0(K')}{\Phi_4^{-1}(K')_2}K'\right)\right)^2}{2\sigma^2}}\mathrm{d}\nu_0.$$

The probability measure P (V)

Theorem 3.11 ([9]). *If ν_0^1 is such a probability measure on $\mathcal{K}_0^1$ for which $\alpha_0(K)^{-1}(\nu_0^1)$ has uniform distribution, $\nu_0 = \Phi_4(\nu_0^1 \otimes \nu)$ where ν is a probability measure on $(0, \infty)$ and Φ is the probability function of the standard normal distribution then*

$$P(\mathcal{A}) := \frac{4p(\mathcal{A})}{\left(\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)\right)} =$$

$$= \frac{4}{\left(\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)\right)\sqrt{2\pi\sigma^2}} \int\limits_{K' \in \mathcal{A}} e^{-\frac{\left(\delta^h\left(B_{E}, \frac{\alpha_0(K')}{\Phi_4^{-1}(K')_2} K'\right)\right)^2}{2\sigma^2}} \mathrm{d}\nu_0$$

is a probability measure on $\mathcal{K}_0$. Moreover $\alpha_0(K)^{-1}(P)$ has truncated normal distribution on the interval $[\frac{1}{2}, 1)$, (with mean $\frac{1}{2}$ and variance $\left(\frac{\sigma}{2}\right)^2$), so

$$\alpha_0(K)^{-1}(P)\left(\left\{\frac{1}{2} \leq t \leq c\right\}\right) = P\left(\{\mathcal{K} \in \mathcal{K}_0 \mid \alpha_0(K) \leq c\}\right) = \frac{\Phi\left(\frac{c - \frac{1}{2}}{\frac{\sigma}{2}}\right) - \Phi(0)}{\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)}.$$

The probability measure P (VI)

For a fixed $r \in \mathbb{N}$ consider a sequence (α_i^r) of positive numbers which holds the property $\sum_{i=1}^{\infty} \alpha_i^r = 1$. Let $L_i^r(l)$ be the i^{th} element of the r-th subset of the above partition of $\mathcal{H}^l_{m(n,k)}$. Thus it is a convex hull of exactly r copies of bodies from $S\left(K^l_{m(n,k)}\right)$. We give it the weight $\frac{\alpha_i^r}{2^r}$.

Definition 3.19 ([9]). *Choose a sequence of positive numbers β_l with again the property $\sum_{l=1}^{\infty} \beta_l = 1$. Define a measure $\widetilde{\widetilde{\mu_0^1}}$ by the equality:*

$$\widetilde{\widetilde{\mu_0^1}} := \lim_{n\to\infty} \sum_{k=1}^{2^{n+1}} \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \sum_{i=1}^{\infty} \frac{\beta_l \alpha_i^r}{2^{n+1+r}} \delta_{L_i^r(l)}.$$

Then we proved the theorem:

Theorem 3.12 ([9]). *On the space of norms there is a probability measure P with the following properties:*

- *The neighborhoods has positive measure.*
- *The set of polytopes has zero measure.*
- *The set of smooth bodies has measure 1.*
- *The pushforward $\alpha_0(K)^{-1}(P)$ of P has truncated normal distribution on the interval $[\frac{1}{2}, 1)$.*

Fundamental theorem on approximation

It is clear that a deterministic time-space model is a special trajectory of the random time-space model. The following theorem is essential.

Theorem 1 *For a trajectory $L(\tau)$ of the random time-space model, for a finite set $0 \leq \tau_1 \leq \cdots \leq \tau_s$ of moments and for a $\varepsilon > 0$ there is a deterministic time-space model defined by the function $K(\tau)$ for which*

$$\sup_i\{\rho_H\left(L(\tau_i), K(\tau_i)\right)\} \leq \varepsilon.$$

The product in a deterministic time-space

Definition 12 *For two vectors $s_1+\tau_1$ and $s_2+\tau_2$ of the deterministic time-space model define their product with the equality*

$$[s_1+\tau_1, s_2+\tau_2]^{+,T} := [s_1, s_2]^{\tau_2} + [\tau_1, \tau_2] =$$

$$= [s_1, s_2]^{\tau_2} - \tau_1\tau_2.$$

Here $[s_1, s_2]^{\tau_2}$ means the s.i.p defined by the norm $\|\cdot\|^{\tau_2}$. This product is not a Minkowski product, as there is no homogeneity property in the second variable. On the other hand the additivity and homogeneity properties of the first variable, the properties on non-degeneracy of the product are again hold,

Imaginary unit sphere

The points of $H^{+,T}$ can be defined by the union

$$\cup\left\{\left\{s+\tau \text{ where } \sqrt{[s,s]^{\tau}+1}=\tau\right\}\ ,\tau\geq 1\right\}.$$

Our assumption on $K(\tau)$ cannot guaranty that for every $s\in S$ there is at least one τ is holding the equality $\sqrt{[s,s]^{\tau}+1}=\tau$. On the other hand if we assume that $\rho_H(K(\tau),B_E)\leq 1$ the ball $2K(\tau)$ contains the Euclidean ball B_E for every τ. Hence $[s,s]^{\tau}\leq 4\|s\|_E^2$ so for all τ with $\tau^2>4\|s\|_E^2+1$, the inequality $[s,s]^{\tau}+1<\tau^2$ holds. Since for a non-zero vector s we have $[s,s]^1+1>1$, the

The de Sitter sphere

The points of the de Sitter sphere $G^{+,T}$ can be defined by the union

$$\cup\left\{\left\{s+\tau e_n \text{ where } \sqrt{[s,s]^{\tau}-1}=\tau\right\}\ ,\ [s,s]^{\tau}\geq 1\right\}.$$

The shape function and the frames

$$\mathrm{K}(x,\tau) : (S, \|\cdot\|_E) \times T \to (S, \|\cdot\|_E)$$

- $\mathrm{K}(x,\tau)$ is homogeneous in its first variable and continuously differentiable in its second one,
- $\mathrm{K}(\{e_1, e_2, e_3\}, \tau)$ is an Auerbach basis of $(S, \|\cdot\|^{\tau})$ for every τ,
- $\mathrm{K}(B_E, \tau) = K(\tau)$

Definition 3.25 ([10]). *The frame* $\{f_1(\tau), f_2(\tau), f_3(\tau), o(\tau)\}$ moves with constant velocity with respect to the time-space *if for every pairs* τ, τ' *in* T^+ *we have*

$$f_i(\tau) = \mathrm{K}(f_i(\tau'), \tau) \text{ for all } i \text{ with } 1 \leq i \leq 3$$

and there are two vectors $O = o_1e_1 + o_2e_2 + o_3e_3 \in S$ *and* $v = v_1e_1 + v_2e_2 + v_3e_3 \in S$ *that for all values of* τ *we have*

$$O(\tau) = \mathrm{K}(O,\tau) + \tau\mathrm{K}(v,\tau).$$

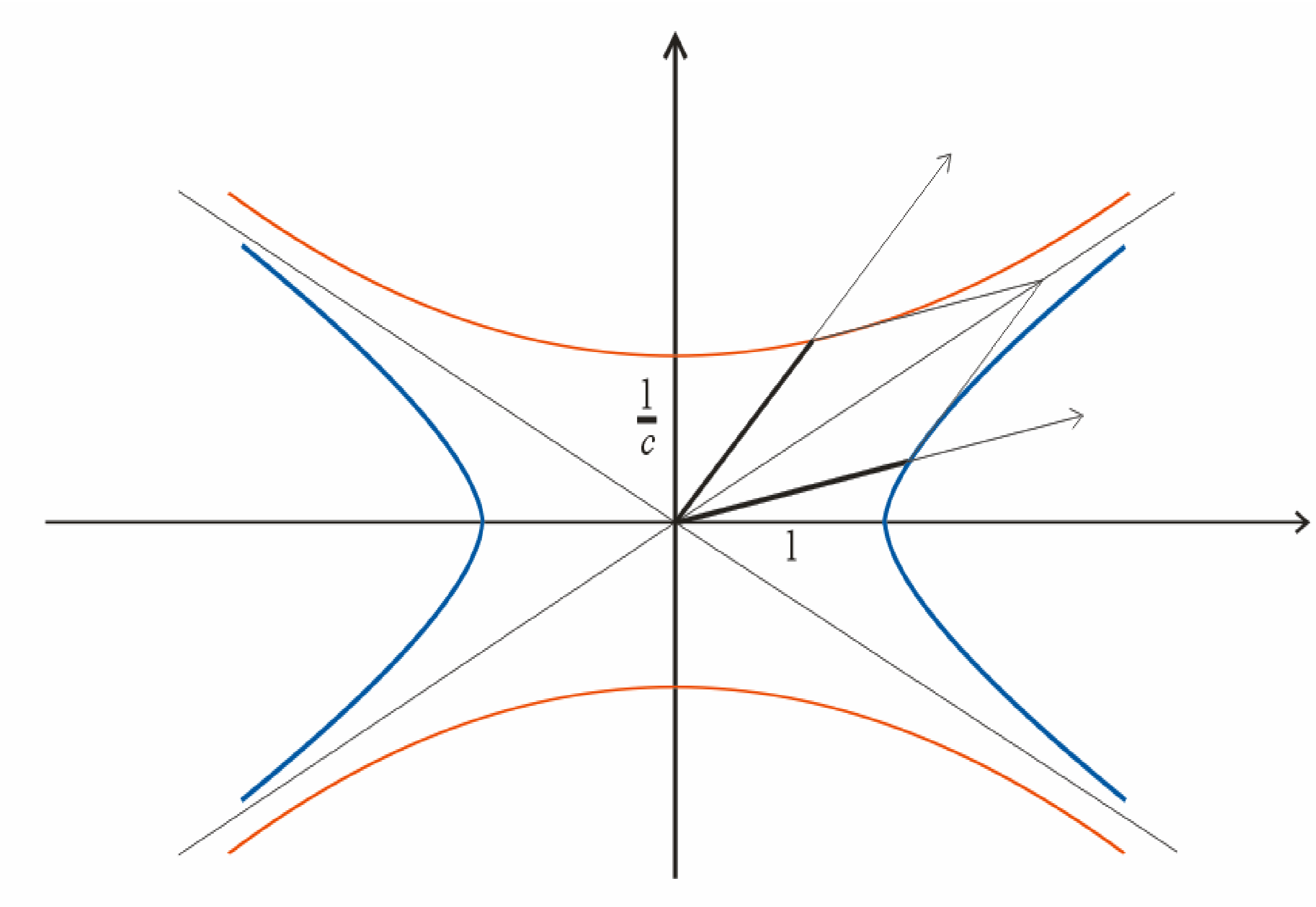
A frame is at rest with respect to the time-space *if the vector* v *is the zero vector of* S.

Definition 3.26 ([10]). *The* time-axis *of the time-space model is the world-line* $O(\tau)$ *of such a particle which moves with constant velocity with respect to the time-space and starts from the origin. More precisely, for the world-line* $(O(\tau),\tau)$ *we have* $\mathrm{K}(O,\tau) = 0$ *and hence with a given vector* $v \in S$,

$$O(\tau) = \tau\mathrm{K}(v,\tau).$$

Lorentz transzformáció

A fény terjedési sebessége vákuumban konstans.



Inerciarendszerek közötti transzformáció a fizikai törvények alakját nem befolyás

On the formulas of special relativity

Axiom 3.1. *The laws of physics are invariant under transformations between frames. The laws of physics will be the same whether you are testing them in frame "at rest", or a frame moving with a constant velocity relative to the "rest" frame.*

Axiom 3.2. *The speed of light in a vacuum is measured to be the same by all observers in frames.*

$$[\mathrm{K}(v,\tau),\mathrm{K}(v,\tau)]^{\tau} = \|v\|_E^2 \qquad \beta = \frac{\|v\|_E}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \boxed{\tau = \gamma\tau_0} \qquad \boxed{L = L_0\sqrt{1-\beta^2} = \frac{L_0}{\gamma}}$$

$$\boxed{\begin{aligned} s &\mapsto \widehat{\mathrm{K}(s',\tau')} = \gamma\left(\mathrm{K}(s,\tau) - \mathrm{K}(v,\tau)\tau\right) \\ \tau &\mapsto \tau' = \gamma\left(\tau - \frac{[\mathrm{K}(s,\tau),\mathrm{K}(v,\tau)]^{\tau}}{c^2}\right), \end{aligned}}$$

$$\begin{aligned} \widehat{\mathrm{K}(s',\tau')} &\mapsto \mathrm{K}(s,\tau) = \gamma\left(\mathrm{K}(s',\tau') + \mathrm{K}(v,\tau')\tau'\right) \\ \tau' &\mapsto \tau = \gamma\left(\tau' + \frac{[\mathrm{K}(s',\tau'),\mathrm{K}(v,\tau')]^{\tau'}}{c^2}\right) \end{aligned}$$

$$\boxed{V(\tau) := \frac{\mathrm{d}S(\tau)}{\mathrm{d}\tau_0} = \gamma\left(\frac{\mathrm{d}(\mathrm{K}(s(\tau),\tau))}{\mathrm{d}\tau} + e_4\right)}$$

$$\boxed{A(\tau) := \frac{\mathrm{d}V}{\mathrm{d}\tau_0} = \gamma(\tau)\frac{\mathrm{d}V}{\mathrm{d}\tau} = \gamma^2(\tau)\frac{\mathrm{d}^2\mathrm{K}(s(\tau),\tau)}{\mathrm{d}\tau^2} + \gamma(\tau)\gamma'(\tau)\frac{\mathrm{d}(\mathrm{K}(s(\tau),\tau))}{\mathrm{d}\tau} + \gamma(\tau)\gamma'(\tau)e_4}$$

Einstein egyenlet (I)

This gives us the equation for a free particle, or the *GEODESIC EQUATION*:

$$\boxed{\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \quad = \quad 0}$$

where $\Gamma^\lambda{}_{\mu\nu}$ is the *AFFINE CONNECTION*:

$$\Gamma^\lambda{}_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu}$$

Note that Γ is symmetric in its lower indices, and, for future reference, it is *not* a tensor.

Lokális inercia rendszerekben a speciális relativitás elmélet formulái érvényesek.

Further note that the proper time interval can be written in terms of dx^μ:

$$\begin{aligned} c^2 d\tau^2 &= \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}dx^\mu\right)\left(\frac{\partial \xi^\beta}{\partial x^\nu}dx^\nu\right) \\ \Rightarrow c^2 d\tau^2 &= g_{\mu\nu} dx^\mu dx^\nu \end{aligned}$$

where $g_{\mu\nu}$ is the *METRIC TENSOR* (note it is symmetric):

$$\boxed{g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}}$$

Einstein egyenlet (II)

$$\boxed{\Gamma^{\sigma}{}_{\lambda\mu} \;=\; \frac{1}{2} g^{\nu\sigma}\left\{\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}\right\}.}$$

$$\begin{aligned}\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} &= \frac{\partial}{\partial x'^{\lambda}}\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}\right) = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial V^{\nu}}{\partial x'^{\lambda}} + \frac{\partial x^{\rho}}{\partial x'^{\lambda}}\frac{\partial^2 x'^{\mu}}{\partial x^{\rho}\partial x^{\nu}} V^{\nu} \\ &= \frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial x^{\rho}}{\partial x'^{\lambda}}\frac{\partial V^{\nu}}{\partial x^{\rho}} + \frac{\partial x^{\rho}}{\partial x'^{\lambda}}\frac{\partial^2 x'^{\mu}}{\partial x^{\rho}\partial x^{\nu}} V^{\nu}\end{aligned}$$

$$\boxed{V^{\mu}{}_{;\lambda} \equiv \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}{}_{\lambda\rho} V^{\rho}}$$

Kovariáns derivált

Görbületi tenzor

$$\boxed{R^{\alpha}{}_{\sigma\rho\beta} \equiv \Gamma^{\alpha}{}_{\beta\sigma,\rho} - \Gamma^{\alpha}{}_{\rho\sigma,\beta} + \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\nu}{}_{\sigma\beta} - \Gamma^{\alpha}{}_{\beta\nu}\Gamma^{\nu}{}_{\sigma\rho}}$$

(Vannak szimmetriái de nem szimmetrikus)

Ricci tenzor

$$R_{\alpha\beta} \;\equiv\; R^{\mu}{}_{\alpha\mu\beta}$$

(szimmetrikus)

Ricci skalár vagy skalár görbület

$$R \;\equiv\; R^{\alpha}{}_{\alpha}$$

$$\boxed{G^{\mu\nu} - \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}}$$

$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ (Einstein tenzor)

$T^{\mu\nu}$ (Energia-momentum tenzor)

Λ (Kozmológiai konstans)

General relativity theory (embedded metrics)

Minkowski-Lorentz metric.

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

$$\boxed{\mathrm{K}\left(v,\tau\right) = v}$$

Friedmann-Lemaître-Robertson-Walker metrics

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{R^2(t)}{1+\frac{1}{4}k(x^2+y^2+z^2)}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$$

$$\boxed{\mathrm{K}\left(v,\tau\right) = \frac{\sqrt{1+\frac{1}{4}k(\tau)\|v\|_E^2}}{R(\tau)}v.}$$

General relativity theory (visualization)

3.2. Three-dimensional visualization of a metric in a four-time-space. The second method is when we consider a four-dimensional time-space and a three-dimensional sub-manifold in it with the property that the metric of the time-space at the points of the sub-manifold can be corresponded to the given one. This method gives a good visualization of the solution in a case when the examined metric has some speciality e.g. there is no dependence on time or (and) the metric has a spherical symmetry. The examples of

Schwarzschild metric.

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2$$

$$\tau = t - \frac{r_s}{2} \ln\left(t + \frac{r_s}{4}\right) + C \qquad \hat{g}(\tau) = t$$

$$\mathrm{K}\left(v, \tau\right) = \frac{\hat{g}(\tau)}{g(\hat{g}(\tau))} v = \left(1 + \frac{r_s}{4\hat{g}(\tau)}\right)^{-2} v.$$

Bertotti-Robinson metric

$$\mathrm{d}s^2 = \frac{Q^2}{r^2}\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \qquad \tau = e \ln r \qquad \mathrm{K}\left(v, \tau\right) := \frac{e^{\frac{\tau}{Q}}}{Q} v$$

Homogeneous time-space-manifold (I)

Definition 3.28 ([11]). *Let $\mathcal{S}$ be the set of linear mappings $\mathbf{K}(v,\tau):\mathbb{E}^3\times\mathbb{R}\longrightarrow\mathbb{E}^3$ holding the properties of a linear shape-function given in Definition 7. Giving for it the natural topology we say that $\mathcal{S}$ is the* space of shape-functions. *If we have a pair of a four-dimensional topological manifold M and a smooth (C^∞) mapping $\mathcal{K}:M\longrightarrow\mathcal{S}$ with the property that at the point $P\in M$ the tangent space is the time-space defined by $\mathcal{K}(P)=\mathbf{K}^P(s,\tau)\in\mathcal{S}$ we say that it is a* time-space-manifold. *The time-space manifold is* homogeneous *if the mapping $\mathcal{K}$ is a constant function.*

Axiom 3.3. *(Equivalence Principle) At any point in a homogeneous time-space manifold it is possible to choose a* locally-inertial frame *in which the laws of physics are the same as the special relativity of the corresponding time-space.*

$$S(\tau)=\mathbf{K}(s(\tau),\tau)+\tau e_4$$

$$V(\tau)=\gamma(\tau)\left(\frac{\mathrm{d}(\mathbf{K}(s(\tau),\tau))}{\mathrm{d}\tau}+e_4\right)=\gamma(\tau)\left(\mathbf{K}(v(\tau),1)+e_4\right)$$

$$A(\tau)=\gamma^2(\tau)\mathbf{K}(a(\tau),0)+\gamma^4(\tau)\frac{[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]'}{c^2}\mathbf{K}(v(\tau),1)+$$
$$+\gamma^4(\tau)\frac{[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^{\prime}}{c^2}e_4,$$

Homogeneous time-space-manifold (II)

$$\frac{\mathrm{d}^2 S'(\tau_0)}{\mathrm{d}\tau_0^2} + \Gamma(S', S)\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0} = 0$$

$$g(S', S)_{\varphi\psi} = {}_{\varphi}{}^{\mu}K \frac{\partial x^{\alpha}}{\partial x'^{\mu}}{}_{\alpha}{}^{\delta}k\eta_{\delta,\varepsilon}k^{\varepsilon}{}_{\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}K^{\nu}{}_{\psi}$$

$$\Gamma(S', S)^{\sigma}{}_{\lambda\mu} = \frac{1}{2}g(S, S')^{\nu\sigma}\left\{\frac{\partial g(S', S)_{\mu,\nu}}{\partial x'^{\lambda}} + \frac{\partial g(S', S)_{\lambda,\nu}}{\partial x'^{\mu}} - \frac{\partial g(S', S)_{\mu,\lambda}}{\partial x'^{\nu}}\right\}$$

Here $R(S', S)^{\alpha}{}_{\sigma\rho\beta}$ is the *Riemann curvature tensor* defined by

$$R(S', S)^{\alpha}{}_{\sigma\rho\beta} := \Gamma(S', S)^{\alpha}{}_{\beta\sigma;\rho} - \Gamma(S', S)^{\alpha}{}_{\rho\sigma;\beta} + \Gamma(S', S)^{\alpha}_{\rho\nu}\Gamma(S', S)^{\nu}_{\sigma\beta} - \Gamma(S', S)^{\alpha}_{\beta\nu}\Gamma(S', S)^{\nu}_{\sigma\rho}.$$

The Ricci Tensor and the scalar curvature defined by

$$R(S', S)_{\sigma\beta} := R(S', S)^{\alpha}{}_{\sigma\alpha\beta} \text{ and } R(S', S) := R(S', S)^{\sigma}{}_{\sigma},$$

$$\boxed{R(S', S)^{\mu\nu} - \frac{1}{2}g(S', S)^{\mu\nu}R(S', S) - \Lambda g(S', S)^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}}$$

Thank you for your attention!