On Infinitesimal Orbit Types

D. Szeghy

Department of Geometry
Eötvös Loránd University

2013. 10. 03.

Actions - Orbits

Definition

Let $\alpha : G \times M \to M$ be a **differentiable action** of a Lie group G on a connected differentiable manifold M, i.e. the map α is differentiable, $\alpha\left(g, \alpha\left(h, x\right)\right) = \alpha\left(g \cdot h, x\right)$ holds for every $x \in M,\ g,\ h \in G$ and $\alpha\left(e, x\right) = x$ for the unit element $e \in G$.

The action $\alpha : G \times M \to M$ induces an action on the tangent bundle $T\alpha : G \times TM \to TM$. If (M, g) is a semi-Riemannian manifold and the induced action $T\alpha$ preserves the metric tensor g then the action α is called isometric.

Definition

Let $G\left(x\right) = \{gx \mid g \in G\}$ denote the **orbit** of x.

Dense orbit by an isometric Riemannian action

Wrapping up $\mathbb{R}^2$ to a torus and the action of $\mathbb{R}$ on it by an irrational translatio

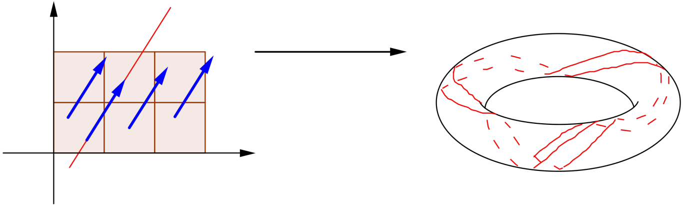

Figure: Dense orbits

Study of the orbits

Natural questions to study

- Similarities and differences
- The action on the orbits
- The action in a neighbourhood of an orbit

Definition

The **isotropy group** of a point x, also called the stabilizer of x, is $G_x \stackrel{def}{=} \{g \in G \mid gx = x\}$.

Definition

The **orbit type** of the orbit $G(x)$ is the conjugacy class $\left\{gG_xg^{-1} \mid g \in G\right\}$.

Principal Orbit Type Theorem

Definition

The orbit type of the orbit $G(x)$ is greater than or equal to that of $G(y)$ if $G_{x'} \subseteq G_{y'}$ for some $x' \in G(x)$ and $y' \in G(y)$.

It is not a partial ordering in general!

Theorem (Principal Orbit Type Theorem M-S-Y)

Let $\alpha : G \times M \to M$ be a differentiable action of a Lie group G on a connected manifold M. If the action is proper or the group is compact, then among the orbit types there is a maximal one, called principal, such that the union of the orbits of this type is an open and dense set in M.

Question

How to exclude compactness or properness?

Dense orbit by an isometric Riemannian action

- The action is not proper
- The group $\mathbb{R}$ is not compact
- But all the orbits have the same type

Wrapping up $\mathbb{R}^2$ to a torus and the action of $\mathbb{R}$ on it by an irrational translatio

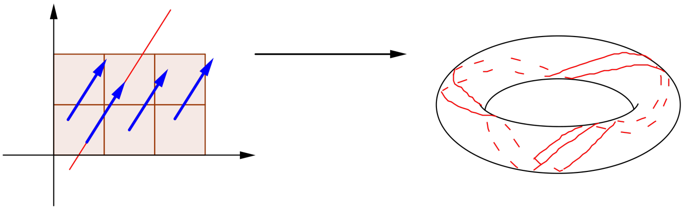

Figure: Dense orbits

POT is not true without compactness in general

The action of $SO(2,1)$ on the Minkowski space $\mathbb{M}^3$ shows that without compactness POT is not true

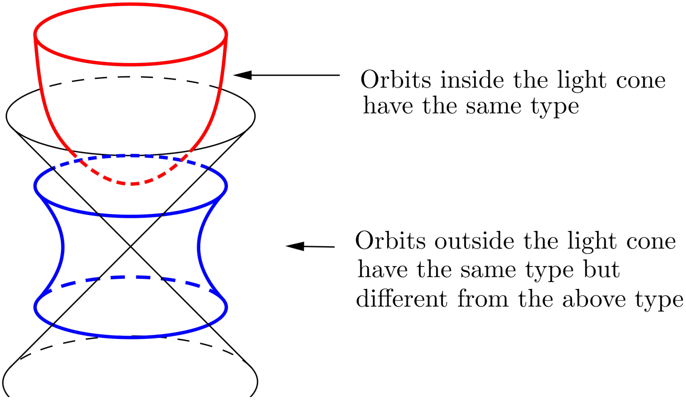

Figure: Example 2

Definition

Let $\alpha : G \times M \to M$ be a differentiable action of a Lie group G on a differentiable manifold M. The infinitesimal orbit type of an orbit $G(x)$ is the whole conjugacy class of the identity component G_x^0 of the isotropy group G_x of x, i.e. the **infinitesimal orbit type** of $G(x)$ is $\{gG_x^0g^{-1} \mid g \in G\}$. This type will be denoted by $\mathrm{inftyp}G(x)$.

- This gives always a partial ordering
- However, example 2 "works" in this case also (inside and outside have different inftype)
- BUT the separating orbit (the light cone) has a special property

Definition

Let $\alpha : G \times M \to M$ be a differentiable action of a Lie group G on a differentiable manifold M. The orbit $G(x)$ is **normalizable** if there is a subspace $\mathcal{N}_x G(x) \subset T_x M$ for which the following holds:

- $\mathcal{N}_x G(x) \oplus T_x G(x) = T_x M$ is a decomposition;
- $\mathcal{N}_x G(x)$ is invariant under the action of G_x.

A *normal bundle* $\mathcal{N} G(x)$ of a normalizable orbit $G(x)$ is a bundle over $G(x)$ obtained from a normal space $\mathcal{N}_x G(x)$ by the action $T\alpha : G \times TM \to TM$, i.e. the **normal bundle** is given by

$$\mathcal{N} G(x) = \cup_{g \in G} T\alpha_g (\mathcal{N}_x G(x)) .$$

Moreover, **the action α is normalizable** if every orbit of α is normalizable.

In Example 2 the separating orbit (the light cone) is non-normalizable!

Dimension problems in general

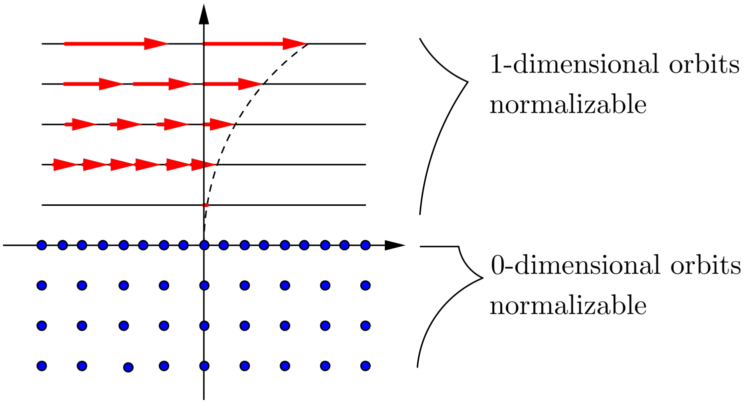

Figure: An action of $\mathbb{R}$ on $\mathbb{R}^2$

Question

How to avoid this problem?

Local model property

Definition

Let $\alpha : G \times M \to M$, $\beta : G \times N \to N$ be differentiable actions of a Lie group G on the differentiable manifolds M and N. A map $\varphi : M \to N$ is **G-equivariant**, if for every $x \in M$, $h \in G$ the equality $\varphi(\alpha(h, x)) = \beta(h, \varphi(x))$ holds.

Definition

Let $\alpha : G \times M \to M$ be a differentiable action of a Lie group G on a differentiable manifold M. Let $G(x)$ be a normalizable orbit and assume that there is a G-invariant neighbourhood $U \subset \mathcal{N}G(x)$ of the zero section and a G-equivariant locally diffeomorphic map $\varphi : U \to M$ for which $\varphi(0_x) = x$. Then we say that $\mathcal{N}G(x)$ is a **local model**.

Why local model property?

The action α is 'difficult' on the manifold

G-equivariant mapping connects these actions

The action $\mathrm{T}\alpha$ is linear on the normal spaces $\mathcal{N}_x G(x)$

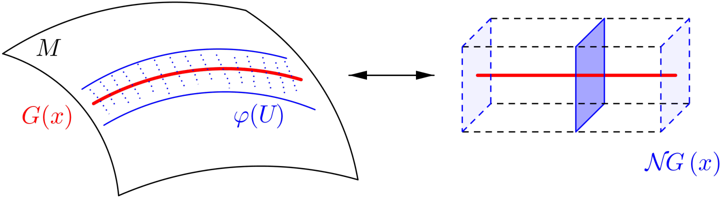

Figure: Local Model Property

An Infinitesimal Principal Orbit Type Theorem

Theorem (IPOT)

Let $\alpha : G \times M \to M$ be a differentiable action of a Lie group G on a connected differentiable manifold M which is normalizable. Assume that every normal bundle is a local model. Then there is a unique maximal infinitesimal orbit type κ, such that the union of the orbits of type κ is an open and dense set in M. Moreover, type κ is stable.

In case of a Riemannian manifold (M, g) and an isometric action on it α, it is easy to see that the orthogonal budle is a normal bundle, and due to the expectational map, the local tube theorem holds everywhere on M. Thus, we have the following generalization of the principal orbit type theorem.

Theorem

If $\alpha : G \times M \to M$ is an isometric action of a Lie group on a connected Riemannian manifold (M, g), then there is a unique maximal infinitesimal orbit type, called infinitesimal principal, and the union of the infinitesimal principal orbits is an open, dense and connected set in M.

Theorem

Let $\alpha : G \times M \to M$ be a differentiable action of a Lie group G on a connected differentiable manifold M and assume, that α is isometric on M with respect some semi-Riemannian metric. Then the maximal dimensional orbits build an open and dense set.

Non-normalizable orbits in the Lorentzian case

Definition

Let (M, g) be a connected Lorentz manifold, $\overline{X} : M \to TM$ a Killing vector field and $z \in M$ such that $\overline{X}(z) \neq 0_z$. If

$$\nabla_{\overline{X}(z)} \overline{X} = \mu \overline{X}(z),$$

holds for some $\mu \in \mathbb{R} - \{0\}$ then the integral curve of $\overline{X}$ through z, which is a pregeodesic, is called a **genuine homogeneous pregeodesic**. If $\mu = 0$ then the integral curve of $\overline{X}$ through z, which is a geodesic, is called a **homogeneous geodesic**.

A geodesic can be both genuine homogeneous pregeodesic and homogeneous geodesic with respect to different Killing fields

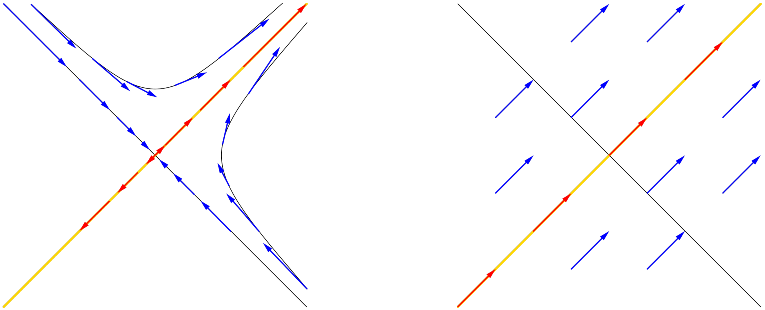

Figure: different Killing fields to the same geodesic

Theorem

If (M, g) is a Lorentz manifold, $\alpha : G \times M \to M$ an isometric action of a Lie group G and the orbit $G(x)$ is non-normalizable, then $G(x)$ is a light-like orbit, such that all the light-like curves of the orbit are light-like homogeneous geodesics or light-like genuine homogeneous pregeodesics, with respect to Killing fields corresponding to the isometric action α.

Theorem

If the connected Lorentz manifold (M, g) is geodesically complete and for every non-normalizable orbit its light-like geodesics are genuine homogeneous pregeodesics, then the union of the normalizable orbits is a dense set in M.

Definition

Let $\alpha : G \times M \to M$ be a differentiable action on a differentiable manifold M. The orbit $G(x)$ is **locally stable** if it has a G-invariant neighbourhood U such that for every $y \in U$ the equality $\operatorname{inftyp} G(x) = \operatorname{inftyp} G(y)$ holds, otherwise $G(x)$ is called **locally unstable**.

Definition

An infinitesimal orbit type κ is called **stable**, if every orbit $G(x)$ with infinitesimal orbit type κ is locally stable. An infinitesimal orbit type η is called **unstable**, if every orbit $G(x)$ with infinitesimal orbit type η is locally unstable.

We can give an isometric action on $\mathbb{M}^3$ where:

- all the orbits are non-normalizable
- all the orbits are locally unstable
- all the orbits have different infinitesimal types
- the action is analytic

Question

Is there hope to extend POT?

Chaotic behavior

We can give an isometric action on $\mathbb{M}^3$ where:

- all the orbits are non-normalizable
- all the orbits are locally unstable
- all the orbits have different infinitesimal types
- the action is analytic

Question

Is there hope to extend POT?

Chaotic behavior

We can give an isometric action on $\mathbb{M}^3$ where:

- all the orbits are non-normalizable
- all the orbits are locally unstable
- all the orbits have different infinitesimal types
- the action is analytic

Question

Is there hope to extend POT?

Chaotic behavior

We can give an isometric action on $\mathbb{M}^3$ where:

- all the orbits are non-normalizable
- all the orbits are locally unstable
- all the orbits have different infinitesimal types
- the action is analytic

Question

Is there hope to extend POT?

Theorem

Let (M, g) be a Lorentz manifold and $\alpha : G \times M \to M$ an isometric action of a Lie group G. Assume that in an open set $U \subset M$, all the points belong to non-normalizable orbits. Then

- *there is no locally stable orbit intersecting U*
- *there are uncountable different infinitesimal orbit types in U*

Theorem

If (M, g) is a Lorentz manifold, $\alpha : G \times M \to M$ an isometric action of a Lie group G then among the maximal dimensional orbits local stability and normalizability is the same, i.e. a maximal dimensional orbit is locally stable if and only if it is normalizable.

Theorem

A special case If (M, g) is a connected, geodesically complete Lorentz manifold, in which there are no conjugate point pairs and $\alpha : G \times M \to M$ is an isometric action of a Lie group G then every infinitesimal obits type is either stable or unstable.

Conjecture

In the case of an isometric action one of the following holds:

- the normalizable orbits build a dense set, therefore, there are countable stable types, such the union of the orbits of these types
- the non-normalizable orbits build an open and dense set, therefore, there uncountable unstable types and the union of the unstable orbits is a dense open set

Thank your for your attention!