The robustness of convex solids

Zsolt Lángi

Department of Geometry, Budapest University of Technology, Hungary

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We deal with convex solids/bodies in $\mathbb{R}^2$ and $\mathbb{R}^3$, with uniform density.
Centre of gravity: defined in the usual way via integrals.
Static equilibrium: when the body rests under gravity on a horizontal plane.

INTRODUCTION

QUESTION

If we want to change the number of static equilibria of a given convex body by a suitable truncation, how large truncation do we need to make? (Largeness is measured, say, as the volume of the truncated part, relative to the full volume of the body)

$$\Downarrow$$

robustness

QUESTION (MODIFIED)

If we want to decrease the number of static equilibria of a given convex body by a suitable truncation, how large truncation do we need to make?

$$\Downarrow$$

downward robustness

FIGURE: Beach with pebbles

Difficulty: there is a coupling between the centre of gravity and the shape $\Rightarrow$ truncation changes the centre of gravity
Solution: decoupling the system

$$\begin{array}{ccc} & \text{robustness} & \\ \swarrow & & \searrow \\ \text{internal} & & \text{external} \end{array}$$

Preliminaries

We distinguish three subclasses of convex bodies.

- $\mathcal{P}_n$: family of n-dimensional convex polytopes,
- $\mathcal{O}_n$: family of n-dimensional convex bodies with smooth (C^∞-class) boundary,
- $\mathcal{K}_n$: family of n-dimensional convex bodies with piecewise smooth boundary.

Definition

For any $K \in \mathcal{K}_2$ and $p \in \operatorname{int} K$, $q \in \operatorname{bd} K$ is an *equilibrium point* of K with respect to p, if the line passing through q and perpendicular to $q - p$, supports K.

Remark

If $\operatorname{bd} K$ is smooth at q, it is equivalent to saying that q is a critical point of the Euclidean distance function $z \mapsto |z - p|$, $z \in \operatorname{bd} K$.

DEFINITION

We call the equilibrium at q *nondegenerate*, if one of the following holds:

- if $\mathrm{bd}\, K$ is smooth at q, then the second derivative of $z \mapsto |z - p|$, $z \in \mathrm{bd}\, K$ at q is not zero,
- if $\mathrm{bd}\, K$ is not smooth at q, then both angles between $p - q$ and one of the two one-sided tangent half lines of $\mathrm{bd}\, K$ at q are acute.

REMARK

If $K \in \mathcal{O}_2$ or $K \in \mathcal{P}_2$, then this definition reduces to the usual concept of nondegeneracy in these classes.

DEFINITION

In the case of a smooth point, we call the nondegenerate equilibrium point q *stable* or *unstable*, if the second derivative of the distance function at q is positive or negative, respectively. In the nonsmooth case, we call the equilibrium point *unstable*.

REMARK

- Poincaré-Hopf Theorem: the numbers of the stable and unstable equilibrium points of any $K \in \mathcal{K}_2$ are equal.
- These two types of points form an alternating sequence in $\mathrm{bd}\, K$.

What about these concepts in $\mathbb{R}^3$?

- equilibrium: in the same way.
- nondegeneracy: in $\mathcal{P}_3 \cup \mathcal{O}_3$ in the natural way .
- There are three types of points: *stable, unstable and saddle*.
- Poincaré-Hopf Theorem:

$$S + U - H = 2.$$

Notation:

- $\{S\}$: family of convex bodies in $\mathcal{K}_2$, with S stable points with respect to their centres of gravity.
- $\{S, U\}$: family of convex bodies in $\mathcal{P}_3 \cup \mathcal{O}_3$, with S stable and U unstable points with respect to their centres.

For $K \in \{S\}$, set

$$\mathcal{F}_<(K) = \{K' \subset K : K' \in \{S'\} \text{ for some } S' < S\}.$$

DEFINITION

Let $K \in \{S\}$. Then we define the downward robustness (or simply *robustness*) of K as the quantity

$$\rho(K) = \frac{\inf\{\text{Area}(K \setminus K') : K' \in \mathcal{F}_<(K)\}}{\text{Area}(K)}.$$

REMARK

In $\mathbb{R}^3$, we may define $\rho(K)$ in an analogous way. Set $\rho_S = \sup\{\rho(K) : K \in \{S\}\}$, $\rho_{S,U} = \sup\{\rho(K) : K \in \{S, U\}\}$.

For $K \in \mathcal{K}_2$ with S stable points with respect to $p \in \operatorname{int} K$, set

$$\mathcal{F}_<(K,p) = \{K' \subset K : K' \in \mathcal{K}_2 \text{ has } S' < S \text{ stable pts wrt } p\}.$$

DEFINITION

Let $K \in \mathcal{K}_2$ have S stable points with respect to p int K. Then we define the downward robustness (or simply *robustness*) of K, with respect to p, as the quantity

$$\rho^{ex}(K,p) = \frac{\inf\{\text{Area}(K \setminus K') : K' \in \mathcal{F}_<(K,p)\}}{\text{Area}(K)}.$$

REMARK

In $\mathbb{R}^3$, we may define $\rho^{ex}(K,p)$ in an analogous way. Set
$\rho^{ex}_S = \sup\{\rho^{ex}(K,G) : K \in \{S\}, G \text{ is the centre of } K\}$,
$\rho^{ex}_{S,U} = \sup\{\rho^{ex}(K,G) : K \in \{S,U\}, G \text{ is the centre of } K\}$.

For $K \in \mathcal{K}_2$ with S stable points with respect to $p \in \operatorname{int} K$, set

$$R(K,p) = \{q \in \mathbb{R}^2 : K \text{ has } S \text{ stable points with respect to } q\}$$

DEFINITION

Let $K \in \mathcal{K}_2$ has S stable points with respect to $p \in \operatorname{int} K$. The *internal robustness* of K with respect to p is

$$\rho^{in}(K,p) = \frac{\min\{|q-p| : q \notin R(K,p)\}}{\operatorname{perim} K}.$$

REMARK

In $\mathbb{R}^3$, we define $\rho^{in}(K,p)$ similarly, by replacing $\operatorname{perim} K$ with $\sqrt{\operatorname{surf} K}$. We set
$\rho^{in}_S = \sup\{\rho^{in}(K,G) : K \in \{S\}, G \text{ is the centre of } K\}$,
$\rho^{in}_{S,U} = \sup\{\rho^{in}(K,G) : K \in \{S,U\}, G \text{ is the centre of } K\}$.

Results: external robustness in the plane

Theorem

Let $K \in \mathcal{K}_2$ contain the origin in its interior, and assume that K has $S \geq 3$ stable points with respect to o. Then

$$\rho^{ex}(K,o) \leq \frac{\tan\frac{\pi}{S} - \frac{\pi}{S}}{S\tan\frac{\pi}{S}},$$

with equality if, and only if K is a regular S-gon and o is its centre.

Corollary

For any $S \geq 3$, we have $\rho^{ex}_S = \frac{\tan\frac{\pi}{S} - \frac{\pi}{S}}{S\tan\frac{\pi}{S}}$, and the convex bodies in $\{S\}$ with maximal external robustness are the regular S-gons.

Theorem

For any $K \in \mathcal{K}_2$ and $p \in \operatorname{int} K$, if K has $S \geq 3$ stable points with respect to p, then

$$\rho^{in}(K,p) \leq \frac{1}{2S},$$

with equality if, and only if, K is a regular S-gon, and p is its centre.

Corollary

For any $S \geq 3$, we have $\rho^{in}_S = \frac{1}{2S}$, and the plane convex bodies $K \in \{S\}$ with maximal internal robustness with respect to their centres of gravity are the regular S-gons.

THEOREM

Let P be a regular polyhedron with S faces, U vertices and $H = S + U - 2$ edges, and let o be the centre of P. Let P' be a convex polyhedron with S faces, U vertices and H edges, each containing an equilibrium point with respect to some fixed $q \in \operatorname{int} P'$. Then

$$\rho^{in}(P', q) \leq \rho^{in}(P, o),$$

with equality if, and only if P' is a similar copy of P, with q as its centre.

REMARK

If, say, P and P' have the same number of stable points, unstable points and edges, then they have the same number of faces, vertices and saddle points.

Example

Let P be a regular tetrahedron of unit surface area with centre o. Truncate P near a vertex, in such a way that does not change the numbers of the three types of equilibria of P, and the truncated part does not intersect the incircle of any face of P, and denote the truncated polyhedron by P'. Then P' has the same numbers of stable, saddle and unstable points with respect to any point of $\text{int}(\rho^{in}(P, o)\mathbf{B})$, but $\text{surf}(P') < \text{surf}(P) = 1$. Thus, $\rho^{in}(P', o) > \rho^{in}(P, o)$.

Results: (full) robustness in 3-space

Definition

By definition, let $\rho(K) = 1$ for any K where the number of equilibria *cannot* be decreased by any trunction.

Remark

We have $\rho(K) = 1$ for any $K \in \{1, 1\}$.

Theorem

We have $\rho_{12} = \rho_{21} = \rho_{22} = 1$.

Question

What if we want to decrease the number of only the stable points, or only that of the unstable points?

Definition

We may define *partial robustness*, i.e. the (relative) volume of a truncation necessary to reduce *either S or U*, (the numbers of stable and unstable points, respectively. We call this *S*-robustness and *U*-robustness, denoted by $\rho^s(K), \rho^u(K)$, respectively.

Clearly, we have $\rho(K) = \min\{\rho^s(K), \rho^u(K)\}$, and also $\rho^s_{1,n} = \rho^u_{n,1} = 1$ for any $n > 2$.

Theorem

If $n > 2$, then $\rho^s_{2,n} = \rho^u_{n,2} = 1$.

Results: partial robustness in 3-space

	U=1	U=2	U=3	U=4
S=1	Gomboc			
S=2		ellipsoids		
S=3				
S=4				regular tetrahedron

Remark

In coastal regions the percentage of pebbles in classes $\{1, n\}, \{n, 1\}$ was found to be below 0.1%.

Remarks

Remark

All our results (and proofs) can be applied for bodies with piecewise C^2-class boundaries.

Remark (Stability of internal/external robustness)

Let $S \geq 3$. For every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, S) > 0$, with $\lim_{\varepsilon \to 0+0} \delta = 0$, such that if $K \in \mathcal{K}_2$ has S stable points with respect to $o \in \operatorname{int} K$, and $\rho^{in}(K, p) > \frac{1}{2S} - \varepsilon$, then the Hausdorff distance of K and a regular S-gon, with o as its centre, is less than δ. This is also true if replace $\rho^{in}(K, o)$ with $\rho^{ex}(K, o)$.

Remark

For every $S \geq 3$, the maximum of $\rho^{in}(K, p)$ over $\mathcal{K}_2$ can be approached by regions with smooth boundaries as well; or in other words, $\frac{1}{2S} = \sup\{\rho^{in}(K, p) : K \in \mathcal{O}_2, p \in \operatorname{int} K\}$.

QUESTIONS

It is known that the first nonempty class in $\mathcal{K}_2$ is $\{2\}$, thus $\rho_2 = 1$.

CONJECTURE

$\rho_n = \rho($regular n-gon$)$ if $n > 2$.

CONJECTURE

The theorem about the internal robustness of platonic solids is valid also for downward external robustness, and for downward full robustness.

We call the equilibrium classes containing platonic solids (classes $\{4,4\}, \{6,8\}, \{8,6\}, \{20,12\}$ and $\{12,20\}$) platonic classes.

PROBLEM

Prove or disprove that in the platonic classes platonic solids have maximal downward full robustness.

Questions; alternative definition for robustness

Conjecture

For $n > 2$, we have $\rho_{2,n} = \rho_{n,2} = \rho_n$.

Conjecture

If $i \geq k$ and $j \geq l$ then $\rho_{i,j} \leq \rho_{k,l}$.

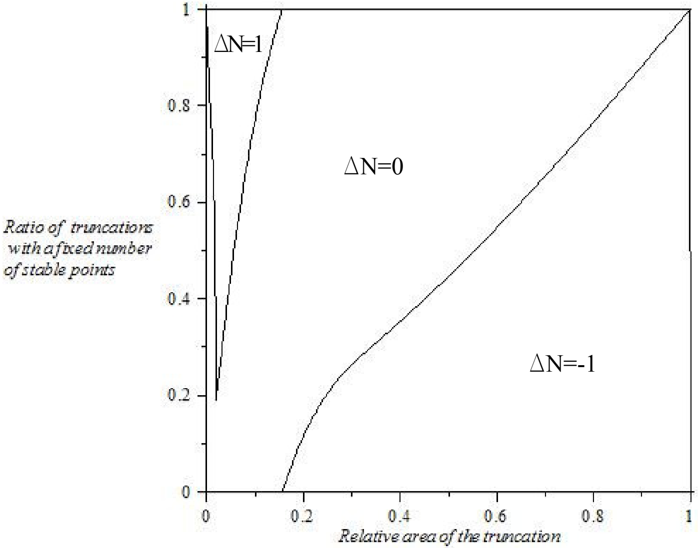
Alternative definition: we may permit only truncations by planes. Advantage:

- numerical experiments are feasible;
- this is the only truncation such that both resulting pieces are convex.

Problem: there is no nice measure on the family of truncations (convex bodies).
Idea: there is a natural measure on the Grassmannian of the hyperplanes, invariant under Euclidean motions. Using direct product, this can be extended to a measure of the space of truncations by finitely many subsequent planes (half spaces).

Given K, its nth order average robustness is the measure of truncations by n planes that does not change the number of equilibria, divided by the measure of truncations by n planes that intersect K.
The average robustness of K is the limit of the nth order average robustness, if it exists.

AVERAGE ROBUSTNESS



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