Finsler felületek, melyek zárt holonómia csoportja $\mathsf{Diff}_{+}^{\infty}(S^1)$

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We developed a method with Zoltán Muzsnay at University of Debrecen for the study of holonomy properties of non-Riemannian Finsler manifolds and obtained that the holonomy group of a large class of projectively flat Finsler manifolds of non-zero constant flag curvature is infinite dimensional.

- Z. Muzsnay, P. T. Nagy, *Finsler manifolds with non-Riemannian holonomy,* Houston J. Math. **38**, (2012), 77–92.
- Z. Muzsnay, P. T. Nagy, *Tangent Lie algebras to the holonomy group of a Finsler manifold,* Communications in Mathematics, **19**, (2011), 137–147.
- Z. Muzsnay, P. T. Nagy, *Projectively flat Finsler manifolds with infinite dimensional holonomy*

M is a 2-dimensional smooth manifold, where smooth means to belong to the C^∞ differentiability class, $\mathfrak{X}^\infty(M)$ is the vector space of smooth vectorfields on M and $\mathsf{Diff}^\infty(M)$ is the group of all C^∞-diffeomorphism of M. (TM, π, M) and (TTM, τ, TM) denote the first and the second tangent bundles of M, respectively.

A *Finsler surface* is a pair $(M, \mathcal{F})$, where $\mathcal{F}\colon TM \to \mathbb{R}$ is a continuous function, smooth on $\hat{T}M := TM \setminus \{0\}$, its restriction $\mathcal{F}_x = \mathcal{F}\big|_{T_xM}$ is a positively homogeneous function of degree one and the symmetric bilinear form

$$g_{x,y}\colon (u,v) \;\mapsto\; g_{ij}(x,y)u^iv^j = \frac{1}{2}\frac{\partial^2 \mathcal{F}_x^2(y + su + tv)}{\partial s\,\partial t}\Big|_{t=s=0}$$

is positive definite at every $y \in \hat{T}_xM$.

Geodesics of $(M, \mathcal{F})$ are determined by a system of 2nd order ordinary differential equation $\ddot{x}^i + 2G^i(x, \dot{x}) = 0$, $i = 1, ..., n$, where $G^i(x, \dot{x})$ are locally given by

$$G^i(x, y) := \frac{1}{4} g^{il}(x, y) \Big(2 \frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y) \Big) y^j y^k. \tag{1}$$

A vectorfield $X(t) = X^i(t) \frac{\partial}{\partial x^i}$ along a curve $c(t)$ is said to be parallel with respect to the associated *homogeneous (nonlinear) connection* if it satisfies

$$D_{\dot{c}} X(t) := \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t), X(t)) \dot{c}^j(t) \Big) \frac{\partial}{\partial x^i} = 0 \text{ with } G^i_j = \frac{\partial G^i}{\partial y^j}. \tag{2}$$

The horizontal distribution $\mathcal{H}TM \subset TTM$ associated to $(M, \mathcal{F})$ is the image of the horizontal lift $X \to X^h : T_x M \to \mathcal{H}_x TM$ defined by $\left(X^i \frac{\partial}{\partial x^i} \right)^h := X^i \left(\frac{\partial}{\partial x^i} - G^k_i(x, y) \frac{\partial}{\partial y^k} \right)$ at each $x \in M$.

Let $\hat{\mathfrak{X}}^\infty(TM)$ be the vector space of smooth sections of the vertical bundle $(\hat{\mathcal{V}}TM, \tau, \hat{T}M)$ over $\hat{T}M := TM \setminus \{0\}$. The *horizontal Berwald covariant derivative* of $\xi \in \hat{\mathfrak{X}}^\infty(TM)$ by $X \in \mathfrak{X}^\infty(M)$ is defined by $\nabla_X \xi := [X^h, \xi]$. In a local coordinate system (x^i, y^i) of TM we denote $G^i_{jk}(x,y) := \frac{\partial G^i_j(x,y)}{\partial y^k}$, then the horizontal Berwald covariant derivative $\nabla_X \xi$ of $\xi(x,y) = \xi^i(x,y)\frac{\partial}{\partial y^i}$ by $X(x) = X^i(x)\frac{\partial}{\partial x^i}$ can be expressed by

$$\left(\frac{\partial \xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial \xi^i(x,y)}{\partial y^k} + G^i_{jk}(x,y)\xi^k(x,y)\right) X^j \frac{\partial}{\partial y^i}. \tag{3}$$

The *Riemannian curvature tensor* field $R_{(x,y)}$ has the expression

$$\frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y)G^i_{km}(x,y) - G^m_k(x,y)G^i_{jm}(x,y).$$

The manifold has constant flag curvature $\lambda \in \mathbb{R}$, if at any point $x \in M$ the local expression of the Riemannian curvature is

$$R^i_{jk}(x,y) = \lambda\left(\delta^i_k g_{jm}(x,y)y^m - \delta^i_j g_{km}(x,y)y^m\right).$$

Projectively flat surfaces

Let (x^1, x^2) be a local coordinate system on M corresponding to the canonical coordinates of the Euclidean space which is projectively related to $(M, \mathcal{F})$. Then the geodesic coefficients are of the form

$$G^i(x, y) = \mathcal{P}(x, y)y^i, \; G^i_k = \frac{\partial \mathcal{P}}{\partial y^k} y^i + \mathcal{P}\delta^i_k,$$

$$G^i_{kl} = \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l} y^i + \frac{\partial \mathcal{P}}{\partial y^k} \delta^i_l + \frac{\partial \mathcal{P}}{\partial y^l} \delta^i_k.$$

where $\mathcal{P}$ is a 1-homogeneous function in y, called the projective factor of $(M, \mathcal{F})$.

Example 1 *(P. Funk) The standard Funk surface* $(\mathbb{D}^2, \mathcal{F})$ *defined by the metric function*

$$\mathcal{F}(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} \pm \frac{\langle x,y\rangle}{1-|x|^2} \tag{4}$$

on the unit ball $\mathbb{D}^2 \subset \mathbb{R}^2$ *is projectively flat with constant flag curvature* $-\frac{1}{4}$*. The projective factor* $\mathcal{P}(x,y)$ *of* $(\mathbb{D}^2, \mathcal{F})$ *can be computed using the formula* $\mathcal{P}(x,y) = \frac{1}{2\mathcal{F}}\frac{\partial \mathcal{F}}{\partial x^i} y^i$*:*

$$\mathcal{P}(x,y) = \frac{1}{2}\,\frac{\pm\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle}{1-|x|^2}. \tag{5}$$

Example 2 *There exists a* 1*-parameter family of projectively flat complete Finsler surfaces* $(S^2, \mathcal{F})$ *of positive curvature with metric function satisfying*

$$\mathcal{F}(0,y) = |y| \cos\alpha, \quad \mathcal{P}(0,y) = |y| \sin\alpha, \quad \textit{with} \quad |\alpha| < \frac{\pi}{2}. \tag{6}$$

in a coordinate neighbourhood centered at $0 \in \mathbb{R}^n$.

R. Bryant introduced and studied this class of Finsler metrics on S^2 with great circles as geodesics. Z. Shen generalized its construction to S^n and obtained this expression.

The holonomy group and the infinitesimal holonomy algebra

We denote by $(\mathcal{I}M, \pi, M)$ the *indicatrix bundle* of the Finsler surface $(M, \mathcal{F})$, the *indicatrix* $\mathcal{I}_x M$ at $x \in M$ is the closed curve

$$\mathcal{I}_x M := \{y \in T_x M;\ \mathcal{F}(y) = 1\} \subset T_x M$$

which is diffeomorphic to the circle S^1. The homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along a curve $c : [0, 1] \to \mathbb{R}$ preserves the value of the Finsler function, hence it induces a map $\tau_c \colon \mathcal{I}_{c(0)}M \longrightarrow \mathcal{I}_{c(1)}M$ between the indicatrices.

The group of diffeomorphisms $\mathsf{Diff}^\infty(\mathcal{I}_x M)$ of the indicatrix $\mathcal{I}_x M$ is a regular infinite dimensional Lie group modeled on the vector space $\mathfrak{X}^\infty(\mathcal{I}_x M)$. Particularly $\mathsf{Diff}^\infty(M)$ is a strong ILB-Lie group. In this category of groups one can define the exponential mapping and the group structure is locally determined by the Lie algebra $\mathfrak{X}^\infty(\mathcal{I}_x M)$ of the Lie group $\mathsf{Diff}^\infty(\mathcal{I}_x M)$ (H. Omori).
The *holonomy group* $\mathsf{Hol}_x(M)$ of the Finsler surface $(M, \mathcal{F})$ at a point $x \in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}^\infty(\mathcal{I}_x M)$ generated by (nonlinear) parallel translations of $\mathcal{I}_x M$ along piece-wise differentiable closed curves initiated at the point $x \in M$.

For any vector fields $X, Y \in \mathfrak{X}^\infty(M)$ on M the vector field $\xi = R(X, Y) \in \mathfrak{X}^\infty(\mathcal{I}M)$ is called a *curvature vector field* of $(M, \mathcal{F})$. If $X, Y \in T_xM$, where $x \in M$, the vector field $y \to R(X, Y)(x, y)$ on $\mathcal{I}_xM$ is a *curvature vector field at* x. The Lie algebra $\mathfrak{R}(M)$ of vector fields generated by the curvature vector fields of $(M, \mathcal{F})$ is called the *curvature algebra* of $(M, \mathcal{F})$. The Lie algebra $\mathfrak{R}_x(M)$ of vector fields generated by the curvature vector fields at x is called the *curvature algebra at* x.

The *infinitesimal holonomy algebra* of $(M, \mathcal{F})$ is the smallest Lie algebra $\mathfrak{hol}^*(M)$ of vector fields on the indicatrix bundle $\mathcal{I}M$ satisfying the following properties

(i) any curvature vector field ξ belongs to $\mathfrak{hol}^*(M)$,

(ii) if $\xi, \eta \in \mathfrak{hol}^*(M)$ then $[\xi, \eta] \in \mathfrak{hol}^*(M)$,

(iii) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}^\infty(M)$ then the horizontal Berwald covariant derivative $\nabla_X \xi$ also belongs to $\mathfrak{hol}^*(M)$.

The restriction $\mathfrak{hol}^*_x(M) := \left\{ \xi\big|_{\mathcal{I}_x M} \; ; \; \xi \in \mathfrak{hol}^*(M) \right\} \subset \mathfrak{X}^\infty(\mathcal{I}_x M)$ of the infinitesimal holonomy algebra to an indicatrix $\mathcal{I}_x M$ is called the *infinitesimal holonomy algebra at the point $x \in M$*. Clearly, $\mathfrak{R}(M) \subset \mathfrak{hol}^*(M)$ and $\mathfrak{R}_x(M) \subset \mathfrak{hol}^*_x(M)$ for any $x \in M$.

Let H be a subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(M)$ of a differentiable manifold M. A vector field $X \in \mathfrak{X}^\infty(M)$ is called *tangent to* $H \subset \mathsf{Diff}^\infty(M)$ if there exists a $\mathcal{C}^1$-differentiable 1-parameter family $\{\Phi(t) \in H\}_{t\in\mathbb{R}}$ of diffeomorphisms of M such that $\Phi(0) = \mathsf{Id}$ and $\frac{\partial \Phi(t)}{\partial t}\Big|_{t=0} = X$. A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{X}^\infty(M)$ is called *tangent to* H, if all elements of $\mathfrak{g}$ are tangent vector fields to H.

The following assertion will be an important tool in the next discussion:

The infinitesimal holonomy algebra $\mathfrak{hol}^*(x)$ *at any point* $x \in M$ *is tangent to the holonomy group* $\mathsf{Hol}(x)$.

1. $\mathsf{Diff}^{\infty}_{+}(S^1)$ and the Fourier algebra

The group $\mathsf{Diff}^{\infty}(M)$ of diffeomorphisms of a compact manifold M is an infinite dimensional Lie group belonging to the class of Fréchet Lie groups. The Lie algebra of $\mathsf{Diff}^{\infty}(M)$ is the Lie algebra $\mathfrak{X}^{\infty}(M)$ of smooth vector fields on M endowed with the negative of the usual Lie bracket of vector fields, the Fréchet Lie group $\mathsf{Diff}^{\infty}(M)$ is modeled on the locally convex topological Fréchet vector space $\mathfrak{X}^{\infty}(M)$. A sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathfrak{X}^{\infty}(M)$ converges to f in the topology of $\mathfrak{X}^{\infty}(M)$ if and only if the functions f_j and all their derivatives converge uniformly to f, respectively to the corresponding derivatives of f.

The difficulty of the theory of Fréchet manifolds is that the inverse function theorem and the existence theorems of differential equations, which are well known for Banach manifolds, are not true in this category. These problems have led to the concept of regular Fréchet Lie groups, the definiton of which requires more careful study (cf. H. Omori, A. Kriegl – P. W. Michor). The basic properties of regular Fréchet Lie groups groups are the existence smooth exponential map from the Lie algebra $\mathfrak{g}$ of G to A and the existence of product integrals, which means the convergence of some approximation methods for solving differential equations. In particular $\mathsf{Diff}^\infty(M)$ is a topological group which is an inverse limit of Lie groups modelled on Banach spaces and hence it is a regular Fréchet Lie group.

Fréchet atlas on the diffeomorphism group $\mathsf{Diff}_+^\infty(S^1)$

Let $S^1 = \mathbb{R} \mod 2\pi\mathbb{Z}$ be the unit circle with the standard counterclockwise orientation and let $q : t \mapsto t \mod 2\pi\mathbb{Z} : \mathbb{R} \to S^1$ be its covering map. The group $\mathsf{Diff}_+^\infty(S^1)$ of orientation preserving diffeomorphisms of the S^1 is the connected component of $\mathsf{Diff}^\infty(S^1)$.

Any $\xi \in \mathfrak{X}^\infty(S^1)$ can be written in the form $\xi(t) = u(t)\frac{d}{dt}$, where $\frac{d}{dt} \in \mathfrak{X}^\infty(S^1)$ is the positively oriented unit tangent vector field on S^1 and $u \circ q$ is a 2π-periodic smooth functions on $\mathbb{R}$. We identify the space $\mathfrak{X}^\infty(S^1)$ with the Fréchet space $C^\infty_{2\pi}(\mathbb{R})$ of 2π-periodic smooth real functions on $\mathbb{R}$.

For $x, y \in S^1$, $s_0 \in q^{-1}(x)$ and $t_0 \in q^{-1}(y) \cap [s_0 - \pi, s_0 + \pi]$ the arc-length distance of x, y in the circle S^1 is $\delta(x, y) = |t_0 - s_0| \leq \pi$. If $\delta(x, y) < \pi$ and $s_0 \in q^{-1}(x)$ then $t_0 \in q^{-1}(y) \cap (s_0 - \pi, s_0 + \pi)$ is uniquely determined and $\delta^+(x, y) := t_0 - s_0$ is the oriented arc-length distance of x, y in S^1.

The universal covering group $\widetilde{\mathsf{Diff}^{\infty}_{+}(S^1)}$ of the group $\mathsf{Diff}^{\infty}_{+}(S^1)$ of monotone increasing diffeomorphisms of $\mathbb{R}$ consists of monotone increasing diffeomorphisms $F \in \mathsf{Diff}^{\infty}_{+}(\mathbb{R})$ satisfying $\frac{dF}{ds}(s) > 0$ and $F(s+2\pi) = F(s) + 2\pi$ for any $s \in \mathbb{R}$. For a given diffeomorphism $\psi \in \mathsf{Diff}^{\infty}_{+}(S^1)$ and its covering diffeomorphism $P \in \widetilde{\mathsf{Diff}^{\infty}_{+}(S^1)}$ we consider the set

$$\mathcal{U}^{\psi} = \{\phi \in \mathsf{Diff}^{\infty}(S^1) : \sup_{x \in S^1} \delta(\psi(x), \phi(x)) < \pi\} \subset \mathsf{Diff}^{\infty}_{+}(S^1)$$

as a coordinate neighbourhood around ψ in $\mathsf{Diff}^{\infty}_{+}(S^1)$.

Any $\phi \in \mathcal{U}^{\psi}$ can be lifted uniquely to a covering diffeomorphism $F^P[\phi] \in \mathsf{Diff}^{\infty}_{+}(S^1)$ satisfying the relations

$$q \circ F^P[\phi] = \phi \circ q, \quad F^P[\phi](s) \in (P(s) - \pi, P(s) + \pi),$$

$$F^P[\phi](s) = P(s) + \delta^+(\psi \circ q(s), \phi \circ q(s)).$$

Hence $w_\phi(s) := \delta^+(\psi \circ q(s), \phi \circ q(s)) = F^P[\phi](s) - P(s)$ is a 2π-periodic smooth function contained in the open set

$$\mathcal{W}^P = \{w \in C^{\infty}_{2\pi}(\mathbb{R});\ \sup_{s \in \mathbb{R}} |w(s)| < \pi,$$

$$\inf_{s \in \mathbb{R}} \frac{d}{ds}(P(s) + w(s)) > -1\} \subset C^{\infty}_{2\pi}(\mathbb{R}).$$

Conversely, for any function $w \in \mathcal{W}^P$ the map $s \mapsto P(s) + w(s)$ belongs to $\widetilde{\mathsf{Diff}}^\infty_+(S^1)$. There exists a diffeomorphism $\phi_w \in \mathsf{Diff}^\infty_+(S^1)$ such that $F^P[\phi_w] = w$ and hence the map $\mathcal{C}^P$ defined by $\mathcal{C}^P : \phi \mapsto w_\phi : \mathcal{U}^\psi \to \mathcal{W}^P$ is bijective. It follows that for any diffeomorphism $\psi \in \mathsf{Diff}^\infty_+(S^1)$ and its covering diffeomorphism $P \in \widetilde{\mathsf{Diff}}^\infty_+(S^1)$ the pair $(\mathcal{U}^\psi, \mathcal{C}^P)$ determines a coordinate chart of $\mathsf{Diff}^\infty_+(S^1)$ around ψ.

Let be $\psi_1, \psi_2 \in \mathsf{Diff}^\infty_+(S^1)$ and let P_1 and P_2 be their covering diffeomorphisms. If ϕ belongs to $\mathcal{U}^{\psi_1} \cap \mathcal{U}^{\psi_2}$ then

$$\mathcal{C}^{P_2}(\phi) = F^{P_2}[\phi] - P_2 = \mathcal{C}^{P_1}(\phi) + (F^{P_2}[\phi] - F^{P_1}[\phi]) + (P_1 - P_2).$$

Since $F^{P_2}[\phi] - F^{P_1}[\phi] = 2\pi h$ with some $h \in \mathbb{Z}$, we obtain $\mathcal{C}^{P_2}(\phi) = \mathcal{C}^{P_1}(\phi) + (P_1 - P_2) + 2\pi h$, i.e. the coordinate transformation $\mathcal{C}^{P_2} \circ {\mathcal{C}^{P_1}}^{-1} : \mathcal{C}^{P_1}(\phi) \mapsto \mathcal{C}^{P_2}(\phi)$ is the translation by the function $(P_1 - P_2) + 2\pi h$, which is clearly a smooth map of the Fréchet space $C^\infty_{2\pi}(\mathbb{R})$.

Fourier and Witt algebra

Any element of $\mathsf{Vect}(S^1)$ can be written in the form $f(t)\frac{d}{dt}$, where f is a 2π-periodic smooth functions on the real line $\mathbb{R}$. A sequence $\{f_j\frac{d}{dt}\}_{j\in\mathbb{N}} \subset \mathsf{Vect}\, S^1$ converges to $f\frac{d}{dt}$ in the Fréchet topology of $\mathsf{Vect}(S^1)$ if and only if the functions f_j and all their derivatives converge uniformly to f, respectively to the corresponding derivatives of f. The Lie bracket on $\mathsf{Vect}(S^1)$ is given by

$$[f\frac{d}{dt}, g\frac{d}{dt}] = (g\frac{df}{dt} - \frac{dg}{dt}f)\frac{d}{dt}.$$

The *Fourier algebra* $\mathsf{F}(S^1)$ on S^1 is the Lie-subalgebra of $\mathsf{Vect}(S^1)$ consisting of vector fields $f\frac{d}{dt}$ such that $f(t)$ has finite Fourier series, i.e. $f(t)$ is a Fourier polynomial. The vector fields $\frac{d}{dt},\ \cos nt\frac{d}{dt},\ \sin nt\frac{d}{dt},\ n \in \mathbb{N},$ provide a basis for $\mathsf{F}(S^1)$. A direct computation shows that the vector fields $\frac{d}{dt},\ \cos t\frac{d}{dt},$ $\sin t\frac{d}{dt},\ \cos 2t\frac{d}{dt},\ \sin 2t\frac{d}{dt}$ generate the Lie algebra $\mathsf{F}(S^1)$. The complexification $\mathsf{F}(S^1)\otimes_{\mathbb{R}}\mathbb{C}$ of $\mathsf{F}(S^1)$ is called the *Witt algebra* $\mathsf{W}(S^1)$ on S^1 having the natural basis $ie^{int}f\frac{d}{dt},\ n \in \mathbb{Z}$, with Lie bracket $[ie^{imt}f\frac{d}{dt}, ie^{int}f\frac{d}{dt}] = ie^{i(n-m)t}f\frac{d}{dt}$.

The *Fourier algebra* $\mathsf{F}(S^1)$ is a dense subalgebra of $\mathsf{Vect}(S^1)$ with respect to the Fréchet topology. This assertion follows from the facts that the Fejér's means of the sequence of partial sums of the Fourier series of a smooth function f converges uniformly to f and that the Fourier series of the derivatives of a smooth function f are the derivatives of the Fourier series of f.

Lemma 1 *The topological closure* $\overline{\exp(\mathsf{F}(S^1))}$ *of the group generated by the exponential image of the Fourier algebra* $\mathsf{F}(S^1)$ *is the orientation preserving diffeomorphism group* $\mathsf{Diff}^\infty_+(S^1)$.

Proof. The exponential mapping is continuous and the Fourier algebra is dense in the Lie algebra $\mathsf{Vect}(S^1)$, hence $\overline{\exp(\mathsf{F}(S^1))}$ contains the normal subgroup generated the exponential image $\exp(\mathsf{Vect}(S^1))$. Since $\mathsf{Diff}^\infty(S^1)$ is a simple group we get $\overline{\exp(\mathsf{F}(S^1))} = \mathsf{Diff}^\infty_+(S^1)$. ∎

Topological closure of the holonomy group

In some cases, the topological closure of the holonomy group may reflect geometric properties of Finsler manifolds of arbitrary dimension.

Proposition 1 *The group generated by the exponential image* $\exp(\mathfrak{hol}^*(x))$ *of the infinitesimal holonomy algebra* $\mathfrak{hol}^*(x)$ *at a point* $x \in M$ *is a subgroup of the topological closure* $\overline{\mathsf{Hol}(x)}$ *of the holonomy group .*

Proof. If $c : \mathbb{R} \to \mathsf{Diff}^\infty_+(M)$ is a smooth 1-parameter family of diffeomorphisms of M with $c(0) = \mathsf{id}$, then the proof of Corollary in [**?**] yields that the sequence $c(\frac{t}{n})^n$ of diffeomorphisms converges to $\exp(t\,\dot c(0))$ uniformly in all derivatives. For any element $X \in \mathfrak{g}$ there exists a $\mathcal{C}^1$-differentiable 1-parameter family $\{\Phi(s) \in H\}_{s\in\mathbb{R}}$ of diffeomorphisms of M such that $\Phi(0) = \mathsf{Id}$ and $\frac{\partial\Phi(s)}{\partial s}\Big|_{s=0} = X$. Then $\lim_{n\to\infty} \Phi\left(\frac{s}{n}\right)^n = \exp sX$. It follows that $\{\exp sX;\ s \in \mathbb{R}\} \subset H$ for any $X \in \mathfrak{g}$. Hence if the Lie subalgebra $\mathfrak{g}$ of $\mathfrak{X}^\infty(M)$ is tangent to a subgroup H of $\mathsf{Diff}^\infty(M)$, then the exponential image $\exp(\mathfrak{g})$ of $\mathfrak{g}$ is contained in the topological closure of the subgrop H. ∎

In the case of Finsler surfaces the indicatrix is diffeomorphic to S^1 at any point $x \in M$, hence the curvature vector fields at $x \in M$ are proportional to any given non-vanishing curvature vector field. The following statement provides a tool to prove the isomorphism of the topological closure of the holonomy group of a Finsler surface with $\mathsf{Diff}^\infty_+(S^1)$.

Theorem 1 *If the infinitesimal holonomy algebra $\mathfrak{hol}^*(x)$ at a point x of a simply connected Finsler surface contains the Fourier algebra $\mathsf{F}(S^1)$ on the indicatrix at x then $\overline{\mathsf{Hol}(x)}$ is isomorphic to $\mathsf{Diff}^\infty_+(S^1)$.*

Proof. Since the Finsler surface is simply connected, $\overline{\mathsf{Hol}(x)} \subset \mathsf{Diff}^\infty_+(S^1)$. From the other hand $\overline{\mathsf{Hol}(x)} \subset \exp(\mathsf{F}(S^1))$. The exponential mapping is continuous and the Fourier algebra is dense in the Lie algebra of $\mathsf{Diff}^\infty(S^1)$, hence $\overline{\mathsf{Hol}(x)}$ contains the normal subgroup generated by the exponential image of $\mathsf{Vect}(S^1)$. Since $\mathsf{Diff}^\infty(S^1)$ is a simple group (cf. [**?**]) we get $\overline{\mathsf{Hol}(x)} = \mathsf{Diff}^\infty_+(S^1)$. ∎

2. Holonomy of projective Finsler surfaces of constant curvature

Theorem 2 *Let* $(M, \mathcal{F})$ *be a simply connected projectively flat Finsler surface of non-zero constant curvature* λ*. Assume that there exists a point* $x_0 \in M$ *such that one of the following conditions hold*

(A) *the induced Minkowski norm* $\mathcal{F}(x_0, y)$ *on* $T_{x_0}M$ *is the euclidean norm* $|y|$,

(B) *the projective factor* $\mathcal{P}(x_0, y)$ *on* $T_{x_0}M$ *satisfies* $\mathcal{P}(x_0, y) = c \cdot |y|$ *with* $0 \neq c \in \mathbb{R}$.

Then the topological closure $\overline{\mathsf{Hol}_{x_0}(M)}$ *of the holonomy group is isomorphic to* $\mathsf{Diff}_+^\infty(S^1)$.

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