Seconda Università degli Studi di Napoli
Dipartimento di Matematica

The Kakeya Problem over Finite Fields

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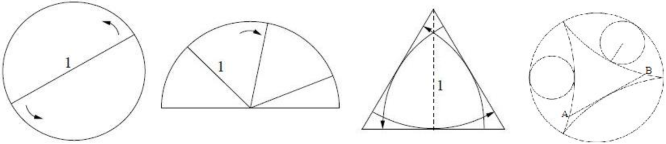
INTRODUCTION

DEFINITION

A Kakeya set in the real plane R^2 is a point set in which a unit line segment can continuously rotate around completely.

PROBLEM (Kakeya Needle Problem)

What is in the real plane the smallest area of a Kakeya set?



The circle of diameter 1 and area $\frac{\pi}{4}$, the semicircle of radius 1 and area $\frac{\pi}{2}$,

the equilateral triangle of height 1 and area $\frac{1}{\sqrt{3}}$,

the deltoid inscribed in a circle of diameter $\frac{3}{2}$ and area $\frac{\pi}{8}$.

The Classical Kakeya Problem

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THEOREM (Besicovitch, 1928)

There exist Kakeya sets in R^2 of arbitrarily small area.

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THEOREM (Besicovitch, 1928)

There exist Kakeya sets in R^2 of arbitrarily small area.

REMARK

In fact, there exist Kakeya sets in R^2 with zero Lebesgue measure. If this is the case, the sets are still necessary two-dimensional, in the sense of Hausdorff dimension (M.Davies, 1971).

The Classical Kakeya Conjecture

DEFINITION

A Kakeya set in R^n is a compact point set containing a unit line segment in every direction.

CONJECTURE

A Kakeya set in R^n has Hausdorff dimension equal to n.

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REMARK

The conjecture is still open for $n > 2$, although many partial results are known (*N.Katz, T.Tao, T.Wolff*).

It was *T.Wolff* who proposed the definition of Kakeya set over finite fields.

REFERENCE

T.Wolff, *Recent work connected with the Kakeya problem.* Prospects in mathematics (Princeton, NJ, 1996), pages 129-162, 1999.

It was *T.Wolff* who proposed the definition of Kakeya set over finite fields.

DEFINITION

A point set E in the $n-$dimensional affine space $AG(n, q)$ is said to to be a Kakeya set if it contains lines in every directions.

REFERENCE

T.Wolff, *Recent work connected with the Kakeya problem.* Prospects in mathematics (Princeton, NJ, 1996), pages 129-162, 1999.

EXAMPLE (Minimal Kakeya Sets)

Let π be the set of all directions of $AG(n,q)$. For every $\alpha \in \pi$ let ℓ_α be a line with slope α. Then a standard Kakeya set is given by

$$E = \bigcup_{\alpha \in \pi} \ell_\alpha.$$

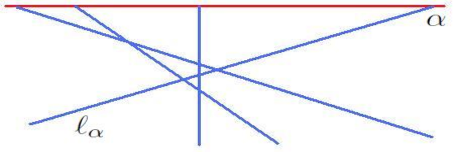

FIGURE: The set E in the case $n = 2$

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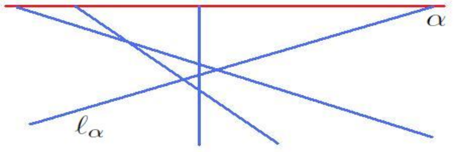

FIGURE: The set E in the case $n = 2$

REMARK

The case $n = 1$ is trivial: the unique Kakeya set is $AG(1,q)$.

The Finite Field Kakeya Problem

PROBLEM

Find the minimum size of a Kakeya set in $AG(n, q)$.

REFERENCE

T.Wolff, *Recent work connected with the Kakeya problem.* Prospects in mathematics (Princeton, NJ, 1996), pages 129-162, 1999.

The Finite Field Kakeya Problem

PROBLEM

Find the minimum size of a Kakeya set in $AG(n, q)$.

REMARK

The problem was firstly raised by T.Wolff in the cited paper.
In the same paper he also made the celebrated finite field Kakeya conjecture.

REFERENCE

T.Wolff, *Recent work connected with the Kakeya problem.* Prospects in mathematics (Princeton, NJ, 1996), pages 129-162, 1999.

The Finite Field Kakeya Conjecture

CONJECTURE

Let E be a Kakeya set in $AG(n,q)$. Then $|E| \geq c_n\, q^n$, where $c_n > 0$ depends only on n.

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The conjecture has had a significant influence in the subject of finite field Kakeya set theory and remained open for more than ten years.

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REMARK

The conjecture has had a significant influence in the subject of finite field Kakeya set theory and remained open for more than ten years.
It was completely solved in 2008 by Z.Dvir using the polynomial method with a beautifully simple argument.

REFERENCE

Z.Dvir, *On the size of Kakeya sets in finite fields*, J. Amer. Math. Soc., 22, 1093-1097, 2009.

THE DVIR'S THEOREM

We recall the following basic fact on polynomials over the field F_q with q elements.

DEFINITION

The polynomials in $F_q[x_1, x_2, ..., x_n]$ of degree at most $q-1$ in each variable and the zero polynomial are called *reduced polynomials*.

PROPOSITION

Let $f \in F_q[x_1, x_2, ..., x_n]$ be a reduced polynomial. If $f(\boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in F_q^n$, then f is the zero polynomial.

PROPOSITION

Let E be a point set in $AG(n,q)$ with $|E| < \binom{n+d}{n}$, for some positive integer d. Then there exists a non zero polynomial $f \in F_q[x_1, x_2, \ldots, x_n]$ of degree $\leq d$ and vanishing on E.

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PROOF

PROPOSITION

Let E be a point set in $AG(n, q)$ with $|E| < \binom{n+d}{n}$, for some positive integer d. Then there exists a non zero polynomial $f \in F_q[x_1, x_2, ..., x_n]$ of degree $\leq d$ and vanishing on E.

PROOF

1. The number of polynomials of degree $\leq d$ is $q^{\binom{n+d}{n}}$.

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PROOF

1. The number of polynomials of degree $\leq d$ is $q^{\binom{n+d}{n}}$.
2. The number of functions from E to F_q is $q^{|E|} < q^{\binom{n+d}{n}}$.

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PROOF

1. The number of polynomials of degree $\leq d$ is $q^{\binom{n+d}{n}}$.
2. The number of functions from E to F_q is $q^{|E|} < q^{\binom{n+d}{n}}$.
3. Let Φ be the map that to any polynomial of degree $\leq d$ associates the restriction to E of its polynomial function. Φ is not one to one.

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3. Let Φ be the map that to any polynomial of degree $\leq d$ associates the restriction to E of its polynomial function. Φ is not one to one.
4. There exist two distinct polynomials g and t of degree $\leq d$ whose polynomial functions assume on E the same values.

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5. $f = g - t$ is a requested polynomial. □

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Let E be a point set in $AG(n,q)$ with $|E| < \binom{n+d}{n}$, for some positive integer d. Then there exists a non zero polynomial $f \in F_q[x_1, x_2, ..., x_n]$ of degree $\leq d$ and vanishing on E.

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2. The number of functions from E to F_q is $q^{|E|} < q^{\binom{n+d}{n}}$.
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5. $f = g - t$ is a requested polynomial. $\square$

PROPOSITION

Let E be a point set in $AG(n,q)$ with $|E| < \binom{n+d}{n}$, for some positive integer d. Then there exists a non zero polynomial $f \in F_q[x_1, x_2, ..., x_n]$ of degree $\leq d$ and vanishing on E.

COROLLARY

Let E be a point set in $AG(n,q)$ and assume that there is no reduced polynomial of degree $\leq d$ vanishing on E. Then

$$|E| \geq \binom{n+d}{n}.$$

PROPOSITION

Let E be a point set in $AG(n,q)$ with $|E| < \binom{n+d}{n}$, for some positive integer d. Then there exists a non zero polynomial $f \in F_q[x_1, x_2, ..., x_n]$ of degree $\leq d$ and vanishing on E.

COROLLARY

Let E be a point set in $AG(n,q)$ and assume that there is no reduced polynomial of degree $\leq d$ vanishing on E. Then

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We are interested in the case $d = q - 1$.

PROPOSITION

Let E be a point set in $AG(n,q)$ with $|E| < \binom{n+d}{n}$, for some positive integer d. Then there exists a non zero polynomial $f \in F_q[x_1, x_2, ..., x_n]$ of degree $\leq d$ and vanishing on E.

COROLLARY ($d = q-1$)

Let E be a point set in $AG(n,q)$ and assume that there is no reduced polynomial of degree less than q vanishing on E. Then

$$|E| \geq \binom{q+n-1}{n}.$$

PROPOSITION

Let E be a Kakeya set in $AG(n,q)$. Then there are no reduced polynomials $f \in F_q[x_1, x_2, ..., x_n]$ of degree $< q$ vanishing on E.

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Assume $f \in F_q[x_1, x_2, ..., x_n]$ is a non zero reduced polynomial of degree $d < q$ vanishing on E.

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PROOF

Assume $f \in F_q[x_1, x_2, ..., x_n]$ is a non zero reduced polynomial of degree $d < q$ vanishing on E. Write:

$f = \sum_{j=0}^{d} f_j$, $d = \deg f \ < \ q$, f_j homogeneous of degree j.

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$f = \sum_{j=0}^{d} f_j$, $d = \deg f \ < \ q$, f_j homogeneous of degree j.

Then the polynomial f_d is non zero, reduced and of degree d.

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$f = \sum_{j=0}^{d} f_j$, $d = \deg f \ < \ q$, f_j homogeneous of degree j.

Then the polynomial f_d is non zero, reduced and of degree d.

- For every vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in F_q^n$, there exist in $AG(n,q)$ a point $\boldsymbol{x_a} = (x_{a1}, x_{a2}, \ldots, x_{an})$ and a line

$$\ell_{\boldsymbol{a}} = \{\boldsymbol{x_a} + t\boldsymbol{a} \ \ : \ \ t \in F_q\}$$

contained in E with slope $\boldsymbol{a}$.

PROPOSITION

Let E be a Kakeya set in $AG(n,q)$. Then there are no reduced polynomials $f \in F_q[x_1, x_2, ..., x_n]$ of degree $< q$ vanishing on E.

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Assume $f \in F_q[x_1, x_2, ..., x_n]$ is a non zero reduced polynomial of degree $d < q$ vanishing on E. Write:
$f = \sum_{j=0}^{d} f_j$, $d = \deg f \; < \; q$, f_j homogeneous of degree j.
Then the polynomial f_d is non zero, reduced and of degree d.

- It follows that $f(\boldsymbol{x}_a + t\boldsymbol{a})$ is a reduced polynomial in the unique variable t vanishing on the q points of ℓ_a . Then $f(\boldsymbol{x}_a + t\boldsymbol{a})$ is the zero polynomial.

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Let E be a Kakeya set in $AG(n, q)$. Then there are no reduced polynomials $f \in F_q[x_1, x_2, ..., x_n]$ of degree $< q$ vanishing on E.

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Assume $f \in F_q[x_1, x_2, ..., x_n]$ is a non zero reduced polynomial of degree $d < q$ vanishing on E. Write:
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Then the polynomial f_d is non zero, reduced and of degree d.

- It follows that $f(\boldsymbol{x_a} + t\boldsymbol{a})$ is a reduced polynomial in the unique variable t vanishing on the q points of $\ell_{\boldsymbol{a}}$. Then $f(\boldsymbol{x_a} + t\boldsymbol{a})$ is the zero polynomial.
- In particular, the coefficient of t^d in $f(\boldsymbol{x_a} + t\boldsymbol{a})$ is zero and it is easy to see that this coefficient is $f_d(\boldsymbol{a})$.

PROPOSITION

Let E be a Kakeya set in $AG(n, q)$. Then there are no reduced polynomials $f \in F_q[x_1, x_2, \ldots, x_n]$ of degree $< q$ vanishing on E.

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Assume $f \in F_q[x_1, x_2, \ldots, x_n]$ is a non zero reduced polynomial of degree $d < q$ vanishing on E. Write:

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Then the polynomial f_d is non zero, reduced and of degree d.

- In particular, the coefficient of t^d in $f(\boldsymbol{x}_a + t\boldsymbol{a})$ is zero and it is easy to see that this coefficient is $f_d(\boldsymbol{a})$.
- It follows that $f_d(\boldsymbol{a}) = 0$, for each $\boldsymbol{a} \in F_q^n$. So, f_d is reduced and identically zero and then it is the zero polynomial, a contradiction.

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THEOREM

Let E be a Kakeya set in $AG(n, q)$. Then

$$|E| \geq \binom{q+n-1}{n} = \frac{1}{n!}\, q(q+1)\cdots(q+n-1) \geq \frac{1}{n!}\, q^n$$

and Wolff's conjecture is true with $c_n = \frac{1}{n!}$.

IMPROVEMENTS OF DVIR'S THEOREM

In 2008, S. Saraf and M. Sudan derived an improvement to the Dvir's constant $c_n = \frac{1}{n!}$.
This was done by considering polynomials that vanish with high multiplicity on a Kakeya set E in $AG(n, q)$. More precisely they proved that

$$|E| \geq \frac{1}{4^n}\, q^n \,.$$

REFERENCE

S. Saraf and M. Sudan, *Improved lower bound on the size of Kakeya sets over finite fields.* Analysis and PDE, 1(3):375-379, 2008.

Improvements of the Saraf-Sudan Constant

In 2009, Z. Dvir, S. Kopparty, S. Saraf, and M. Sudan derived an improvement to the previous Saraf-Sudan constant $c_n = \frac{1}{4^n}$. This was done by considering more sophisticated arguments about polynomials that vanish with high multiplicity on a Kakeya set E in $AG(n, q)$. They were able to prove that

$$|E| \geq \frac{1}{2^n}\, q^n \, .$$

REFERENCE

Z. Dvir, S. Kopparty, S. Saraf, and M. Sudan, *Extensions to the method of multiplicities, with applications to Kakeya sets and mergers*, in FOCS 09 (to appear), 2009.

About the Sharpness

The problem of finding the exact value of the minimum size of a Kakeya set seems to be very hard and gets more difficult as the dimension n increases.

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The problem of finding the exact value of the minimum size of a Kakeya set seems to be very hard and gets more difficult as the dimension n increases.

At this moment, it is completely solved only in dimension two and we will give a brief account of this.

The Finite Field Kakeya Problem: the case $n = 2$, q even

EXAMPLE (Kakeya Sets of Hyperoval type)

Assume q is even and consider in $PG(2,q) = AG(2,q) \cup \ell_\infty$ a dual hyperoval $\mathcal{H}$ containing ℓ_∞. For every point $P \in \ell_\infty$, let ℓ_P the line of $\mathcal{H}$ on P other than ℓ_∞. Then the Kakeya set in $AG(2,q)$

$$E = \Big(\bigcup_{P \in \ell_\infty} \ell_P \Big) \setminus \ell_\infty$$

is of size $\frac{q(q+1)}{2}$.

PROPOSITION

In $AG(2,q)$ with q even, $|E| \geq \frac{q(q+1)}{2}$ for every Kakeya set E. The equality holds iff E is of the hyperoval type.

EXAMPLE (Kakeya Sets of Oval type)

Assume q is odd and consider in $PG(2, q) = AG(2, q) \cup \ell_\infty$ a dual oval $\mathcal{O}$. Every point P on ℓ_∞, but one, belongs to a second line $\ell_P \in \mathcal{O}$ other than ℓ_∞. If A is this remaining point on ℓ_∞, let ℓ_A be a line through it different from ℓ_∞. Then the Kakeya set in $AG(2, q)$

$$E = (\bigcup_{P \in \ell_\infty} \ell_P) \setminus \ell_\infty$$

is of size

$$\frac{q(q+1)}{2} + \frac{q-1}{2}$$

The Finite Field Kakeya Problem: the case $n = 2$, q odd

CONJECTURE (X.Faber Conjecture, 2006)

If q is odd and E is a Kakeya set in $AG(2, q)$, then

$$|E| \geq \frac{q(q+1)}{2} + \frac{q-1}{2}$$

The equality holds if and only if E is of oval type.

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The equality holds if and only if E is of oval type.

PROPOSITION (A.Blokhuis - F.M., 2008)

The Faber's conjecture is true.

REFERENCE

A.Blokhuis and F.M., *The Finite Field Kakeya Problem,* Bridges Between Mathematics and Computer Science, Bolyay Society Mathematical Studies, Vol.19, Grötschel M., Katona G. ((Eds.), Springer, 2008.

The Finite Field Kakeya Problem in the case $n = 2$, q even: the second smallest size

EXAMPLE (Kakeya Sets of quasi-hyperoval type)

Assume q is even and consider in $PG(2, q) = AG(2, q) \cup \ell_\infty$ a Kakeya set $E(\mathcal{H})$ associated to a dual hyperoval $\mathcal{H}$ containing ℓ_∞ :

$$E(\mathcal{H}) = \Big(\bigcup_{P \in \ell_\infty} \ell_P \Big) \setminus \ell_\infty.$$

Fix a point $A \in \ell_\infty$ and a line ℓ' through A different from ℓ_A and ℓ_∞. Then the Kakeya set in $AG(2, q)$

$$E = (E(\mathcal{H}) \setminus \ell_A) \cup (\ell' \setminus \ell_\infty)$$

is of size

$$\frac{q(q+2)}{2}.$$

The Finite Field Kakeya Problem in the case $n = 2$, q even: the second smallest size

PROPOSITION (A. Blokhuis, A.A. Bruen, 1989)

There are no Kakeya sets E in $AG(2, q)$, q even, with

$$\frac{1}{2}q(q+1) < |E| < \frac{1}{2}q(q+2).$$

Furthermore, all Kakeya sets of size $\frac{1}{2}q(q+2)$ are of quasi-hyperoval type.

REFERENCE

A. Blokhuis and A.A. Bruen, *The minimal number of lines intersected by a set of q + 2 points, blocking sets, and intersecting circles,* J. Combin. Theory Ser. A, 50, 308-315, 1989.

Now we describe a Kakeya set, which we will see to be the third smallest example, provided that it exists.

EXAMPLE (Kakeya Sets of $(0, 2, 4)-$arc type)

Let $\mathcal{A}$ be a dual $(q+4)$-arc of type $(0,2,4)$ in $PG(2,q)$, and let $\ell_0, \ell_1, \ell_2, \ell_\infty$ be four concurrent lines of $\mathcal{A}$. Consider the affine plane $AG(2,q) = PG(2,q) \setminus \ell_\infty$. Let $\mathcal{A}'$ be the line set $\mathcal{A} \setminus \{\ell_1, \ell_2\}$. Consider the set

$$E(\mathcal{A}, \ell_1, \ell_2) = \bigcup_{L \in \mathcal{A}'} (L \setminus \ell_\infty)\,.$$

This is a Kakeya set since there is precisely one line of $E(\mathcal{A}, L_1, L_2)$ through every point of L_∞. It has size $\frac{1}{2}q(q+2) + \frac{1}{4}q$.

The Finite Field Kakeya Problem in the case $n = 2$, q even: the third smallest size

PROPOSITION (A. Blokhuis, M. De Boeck, F.M. and L. Storme, 2011)

There are no Kakeya sets E in $AG(2, q)$, q even, with

$$\frac{1}{2}q(q+2) < |E| < \frac{1}{2}q(q+2) + \frac{1}{4}q.$$

Furthermore, all Kakeya sets of size $\frac{1}{2}q(q+2) + \frac{1}{4}q$ are of $(0, 2, 4)-$arc type.

REFERENCE

A. Blokhuis, M. De Boeck, F.M. and L. Storme, *The Kakeya problem: a gap in the spectrum and classification of the smallest examples*, submitted, 2011.

APPLICATIONS

Dual Blocking Sets

DEFINITION

A *dual blocking set* S in $\pi_q = PG(2,q)$ is a point set meeting every blocking set and containing no lines.

REFERENCE

P.J.Cameron, F.M., R.Meshulam, *Dual blocking sets in projective and affine planes,* Geometriae Dedicata, 27, 1988, n.2, 203-207.

Dual Blocking Sets

DEFINITION

A *dual blocking set* S in $\pi_q = PG(2, q)$ is a point set meeting every blocking set and containing no lines.

PROPOSITION

Let S be a minimal dual blocking set in π_q. Then one of the two following possibilities occur:

(i) $S = (\bigcup_{P \in \ell} \ell_P) \setminus \ell$ *is a Kakeya set;*

(ii) $S = \pi_q \setminus (\ell \cup m)$ *is the complement of the union of two distinct lines ℓ and m.*

REFERENCE

P.J.Cameron, F.M., R.Meshulam, *Dual blocking sets in projective and affine planes,* Geometriae Dedicata, 27, 1988, n.2, 203-207.

Dual Blocking Sets

PROPOSITION (1988)

Let S be a dual blocking set in π_q. Then

$$|S| \geq \frac{q(q+1)}{2}.$$

Equality holds if and only if either

(i) *S is the Kakeya set associated to a dual hyperoval and one of its lines; or*

(ii) *$q = 3$ and S is the complement of the union of two distinct lines.*

PROPOSITION

Let S be a dual blocking set in π_q, q odd. Then

$$|S| \geq \frac{q(q+1)}{2} + \frac{q-1}{2}$$

and equality holds if and only if S is a Kakeya set of oval type.

The End

GRAZIE !