Poisson-Lie interpretation of trigonometric Ruijsenaars duality

László Fehér, KFKI RMKI, Budapest and University of Szeged

Talk based on joint work with Ctirad Klimčík, IML, Marseille

• Two integrable many-body systems are dual to each other if the action variables of system (i) are the particle positions of system (ii), and vice versa. Underlying phase spaces are symplectomorphic.

• First example is the self-duality of the rational Calogero system. Interpreted in terms of symplectic reduction by Kazhdan, Kostant and Sternberg (1978).

• Duality was discovered and explored by Ruijsenaars (1988-95) in his direct construction of action-angle variables for Calogero-Sutherland type systems and their 'relativistic' deformations.

A 'dual pair' of integrable many-body systems

Hyperbolic Sutherland system (1971):

$$H_{\text{hyp}-\text{Suth}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{x^2}{2}\sum_{j\neq k}\frac{1}{\sinh^2(q_j - q_k)}$$

Basic Poisson brackets: $\{q_i, p_j\} = \delta_{i,j}$, x: non-zero, real constant.

Rational Ruijsenaars-Schneider system (1986):

$$H_{\text{rat}-\text{RS}}(\hat{p},\hat{q}) = \sum_{k=1}^{n} \cosh(\hat{q}_k)\prod_{j\neq k}\left[1 + \frac{x^2}{(\hat{p}^k - \hat{p}^j)^2}\right]^{\frac{1}{2}}$$

Poisson brackets: $\{\hat{p}_i, \hat{q}_j\} = \delta_{i,j}$ ($\hat{p}_i$ are RS 'particle positions').

Systems describe n 'particles' moving on the line, and are integrable.

Ruijsenaars (1988) constructed 'duality symplectomorphism' (action-angle map) between underlying phase spaces.

Local description of two other dual pairs

Standard trigonometric Ruijsenaars-Schneider [86] system:

$$H_{\text{trigo-RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j\neq k} \left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

It is a relativistic generalization (here with $c = 1$) of

$$H_{\text{trigo-Suth}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{x^2}{2}\sum_{j\neq k} \frac{1}{\sin^2(q_k - q_j)}$$

The dual systems (Ruijsenaars [88,95]):

$$\widehat{H}_{\text{trigo-RS}} = \sum_{k=1}^{n} (\cos \hat{q}_k) \prod_{j\neq k} \left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_k - \hat{p}_j)}\right]^{\frac{1}{2}}$$

$$\widetilde{H}_{\text{rat-RS}} = \sum_{k=1}^{n} (\cos \hat{q}_k) \prod_{j\neq k} \left[1 - \frac{x^2}{(\hat{p}_k - \hat{p}_j)^2}\right]^{\frac{1}{2}}$$

$H_{\text{trigo-RS}}$, $\widehat{H}_{\text{trigo-RS}}$: different real forms of complex trigo RS.

Three self-dual systems

Rational Calogero system:

$$H_{\mathsf{Cal}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{(q_k - q_j)^2}$$

Hyperbolic Ruijsenaars-Schneider system:

$$H_{\mathsf{hyp-RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j\neq k} \left[1 + \frac{\sinh^2 x}{\sinh^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

Compactified trigonometric RS ($\mathrm{III_b}$) system, locally given by

$$H_{\mathsf{compact-RS}} = \sum_{k=1}^{n} (\cos p_k) \prod_{j\neq k} \left[1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

Duality from symplectic reduction: the basic idea

Start with 'big phase space', of group theoretic origin, equipped with *two* commuting families of 'canonical free Hamiltonians'.

Apply suitable *single* symplectic reduction to the big phase space and construct *two* 'natural' models of the reduced phase space.

The two families of 'free' Hamiltonians turn into interesting **many-body Hamiltonians** and **particle-position variables** in terms of *both* models. Their rôle is *interchanged* in the two models.

The natural symplectomorphism between the two models of the reduced phase space yields the 'duality symplectomorphism'.

Motivated by KKS [78], the above 'scenario' was described by Gorsky and Nekrasov in the nineties (see e.g. Fock-Gorsky-Nekrasov-Roubtsov [2000]). They focused on local questions working mostly with infinite-dimensional phase spaces and in a complex holomorphic setting.

Simplest example: Take $T^*(\mathrm{i}u(n)) \simeq \mathrm{i}u(n) \times \mathrm{i}u(n) := \{(X, Y)\}$ for the big phase space. Consider the 'canonical free Hamiltonians' $\mathrm{tr}\,(X^k)$ and $\mathrm{tr}\,(Y^k)$. Reduce by the adjoint action of $U(n)$ choosing the value of the moment map $J(X, Y) = [X, Y]$ from a minimal coadjoint orbit. This yields the self-dual rational Calogero system (OP [76], KKS [78]).

Our purpose is to derive all of Ruijsenaars' dualities by reductions of suitable finite-dimensional phase spaces. Then study new cases: systems with two types of particles, $BC(n)$ systems etc.

• Today, I first explain that the standard trigonometric Ruijsenaars-Schneider system is a symplectic reduction of a natural Poisson-Lie symmetric system on the Heisenberg (symplectic) double of $U(n)$. Generalizes the KKS [78] treatment of the Sutherland system as a reduction of the free particle moving on $U(n)$.

• Then I describe how the Ruijsenaars dual of this system arises in the same reduction procedure.

The phase space of the trigo RS system is $P := T^*Q(n)$, where $Q(n) := \mathbb{T}_n^0/S_n$ with $\mathbb{T}_n^0$ being the regular part of maximal torus $\mathbb{T}_n < U(n)$. The Lax matrix L and symplectic form ω are:

$$L_{jk}(q,p) = \frac{e^{p_k}\sinh(-x)}{\sinh(\mathrm{i}q_j - \mathrm{i}q_k - x)} \prod_{m\neq j}\left[1 + \frac{\sinh^2 x}{\sin^2(q_j - q_m)}\right]^{\frac{1}{4}} \prod_{m\neq k}\left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_m)}\right]^{\frac{1}{4}}$$

$$\omega = \sum_k dp_k \wedge dq_k, \qquad p_k \in \mathbb{R}, \qquad 0 \leq q_k < \pi, \quad q_1 > q_2 > \cdots > q_n$$

The dual system can be *locally* characterized by

$$\hat{L}_{jk}(e^{\mathrm{i}\hat{q}}, \hat{p}) = \frac{e^{\mathrm{i}\hat{q}_k}\sinh(-x)}{\sinh(\hat{p}_j - \hat{p}_k - x)} \prod_{m\neq j}\left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_j - \hat{p}_m)}\right]^{\frac{1}{4}} \prod_{m\neq k}\left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_k - \hat{p}_m)}\right]^{\frac{1}{4}}$$

$\hat{p} = \mathrm{diag}(\hat{p}_1, \ldots, \hat{p}_n) \in \mathfrak{C}_x := \{\hat{p}\,|\,\hat{p}_j - \hat{p}_{j+1} > |x|, \ \ j = 1, \ldots, (n-1)\}$
$e^{\mathrm{i}\hat{q}} \in \mathbb{T}_n$ with $\hat{q} = \mathrm{diag}(\hat{q}_1, \ldots, \hat{q}_n)$. Dual phase space $\hat{P} = \mathbb{T}_n \times \mathfrak{C}_x$ is open submanifold of cotangent bundle of $\mathbb{T}_n$, with $\hat{\omega} = d\hat{p}_k \wedge d\hat{q}_k$.

- The commuting flows associated with $\hat{L}$ are **not** complete on $\hat{P}$.
- $\hat{P}$ is symplectomorpic **(only)** to a dense, open submanifold of P.

Hence $\hat{P}$ needs to be extended, as performed by Ruijsenaars [95].

Reminder on the Kazhdan-Kostant-Sternberg reduction

Consider cotangent bundle $T^*U(n)$ of $U(n)$ (in right-trivialization):

$$T^*U(n) = \{(g, J_L) \,|\, g \in U(n),\ J_L \in u(n)^* \simeq u(n)\}$$

It carries the natural symplectic form

$$\Omega(g, J_L) = d\,\mathrm{tr}\,(J_L dg g^{-1})$$

and two sets of 'canonical free Hamiltonians' $\{h_k\}$ and $\{\widehat{h}_{\pm k}\}$

$$h_k(g, J_L) := \mathrm{tr}\,(\mathrm{i}J_L)^k, \quad \widehat{h}_k(g, J_L) := \Re\mathrm{tr}\,(g^k), \quad \widehat{h}_{-k}(g, J_L) := \Im\mathrm{tr}\,(g^k)$$

- One can write down their Hamiltonian flows explicitly.

- They are invariant under the adjoint action of $U(n)$ on $T^*U(n)$.

Philosophy:

Interesting systems are reductions of 'obviously integrable' systems.

The adjoint action of $U(n)$ on the phase space

$$\eta \triangleright (g, J_L) = (\eta g \eta^{-1}, \eta J_L \eta^{-1}) \qquad \forall \eta \in U(n)$$

is generated by the moment map $J : T^*U(n) \to u(n)^*$ given by

$$J(g, J_L) = J_L + J_R \quad \text{with} \quad J_R(g, J_L) := -g^{-1} J_L g.$$

J is sum of moment maps generating left/right multiplication.

With arbitrary real $x \neq 0$, define $\mu(x) \in u(n)$ by

$$\mu(x)_{jj} = 0, \quad \forall j, \qquad \mu(x)_{jk} = \mathrm{i}x, \quad \forall j \neq k.$$

KKS [78] showed that the moment map constraint

$$J = \mu(x)$$

produces the trigonometric Sutherland system from the Hamiltonian system describing the free particle on $U(n)$: $(T^*U(n), \Omega, h_2)$. The Hamiltonians $\{h_k\}$ give action variables of Sutherland system (and $\{\widehat{h}_{\pm k}\}$ become in effect the Sutherland position variables).

Using another model of the reduced phase space, $\{\widehat{h}_{\pm k}\}$ yield the commuting Hamiltonians of the Ruijsenaars dual of the Sutherland system (and $\{h_k\}$ become in effect the dual position variables).

Poisson-Lie analogue of Kazhdan-Kostant-Sternberg reduction

According to Semenov-Tian-Shansky [85] and Lu-Weinstein [90]:

- P-L analogue of $T^*U(n)$ is Heisenberg double of Poisson $U(n)$.

- The Heisenberg double has 'canonical commuting Hamiltonians'.

As described explicitly by Klimčík [06]:

- Adjoint action (moment map) generalizes to quasi-adjoint action.

We asked:

- **What is the correct moment map value to choose?**

If this is known, the rest is in principle straightforward calculation.

Symplectic structure of Heisenberg double

The Heisenberg double of $U(n)$ is the *real* manifold $GL(n,\mathbb{C})$.

Every $K \in GL(n,\mathbb{C})$ admits two Iwasawa decompositions:

$$K = b_L g_R^{-1} \quad \text{and} \quad K = g_L b_R^{-1} \quad \text{with} \quad g_{L,R} \in U(n),\ b_{L,R} \in B$$

B: group of upper triangular matrices with positive diagonal entries

Define maps $\Lambda_{L,R} : GL(n,\mathbb{C}) \to B$ and $\Xi_{L,R} : GL(n,\mathbb{C}) \to U(n)$

$$\Lambda_{L,R}(K) := b_{L,R} \quad \text{and} \quad \Xi_{L,R}(K) := g_{L,R}$$

$GL(n,\mathbb{C})$ has natural symplectic form (Alekseev-Malkin [94])

$$\omega_+ = \frac{1}{2}\Im \mathrm{tr}\,(d\Lambda_L \Lambda_L^{-1} \wedge d\Xi_L \Xi_L^{-1}) + \frac{1}{2}\Im \mathrm{tr}\,(d\Lambda_R \Lambda_R^{-1} \wedge d\Xi_R \Xi_R^{-1})$$

The Poisson bracket on $(GL(n,\mathbb{C}),\omega_+)$

For any $\Phi_1, \Phi_2 \in C^\infty(GL(n,\mathbb{C}))$:

$$\{\Phi_1,\Phi_2\}_+ = \Im \mathrm{tr}\left(\nabla^R\Phi_1 \rho(\nabla^R\Phi_2) + \nabla^L\Phi_1\rho(\nabla^L\Phi_2)\right)$$

where $\rho := \frac{1}{2}(\pi_{u(n)} - \pi_{\mathcal{B}})$ belongs to the splitting $gl(n,\mathbb{C}) = u(n) + \mathcal{B}$ and we use $gl(n,\mathbb{C})$-valued derivatives

$$\left.\frac{d}{ds}\right|_{s=0} \Phi(e^{sX}Ke^{sY}) = \Im \mathrm{tr}\,(X\nabla^L\Phi(K) + Y\nabla^R\Phi(K)) \quad \forall X, Y \in gl(n,\mathbb{C})$$

Iwasawa maps $\Xi_{L,R} : GL(n,\mathbb{C}) \to U(n)$ and $\Lambda_{L,R} : GL(n,\mathbb{C}) \to B$ are **Poisson maps** if $U(n)$ and B are equipped with their standard Poisson structures. In particular, $\{\ ,\ \}_+$ closes on

$$\Xi^*_{L,R}C^\infty(U(n)) \quad \text{and on} \quad \Lambda^*_{L,R}C^\infty(B)$$

Induced Poisson bracket on $U(n)$ is standard Sklyanin bracket

[defined by Drinfeld-Jimbo r-matrix, $R^{\mathrm{i}} \in \mathrm{End}(u(n))$, $R^{\mathrm{i}}(X) = \pi_{u(n)}(-\mathrm{i}X)$]

Commuting Hamiltonians from dual P-L groups

$C^\infty(U(n))^{U(n)}$: the adjoint (conjugation) invariant functions

$C^\infty(B)^c \equiv C^\infty(B)^{U(n)}$: the center of the Poisson bracket on $C^\infty(B)$
provided by the dressing invariants
$/U(n)$ acts on B according to $\mathrm{Dress}_g(b) := \Lambda_L(gb)/$

$$\Lambda_L^* C^\infty(B)^c = \Lambda_R^* C^\infty(B)^c \quad \text{and} \quad \Xi_R^* C^\infty(U(n))^{U(n)}$$

form **Abelian subalgebras** in $C^\infty(GL(n,\mathbb{C}))$ w.r.t. $\{\ ,\ \}_+$

The commuting Hamiltonians of the dual pair of Ruijsenaars systems will arise from the above two Abelian algebras.

Hence Ruijsenaars duality is linked to Poisson-Lie duality.

Formulae of induced Poisson bracket on B and on $U(n)$

For any $f_1, f_2 \in C^\infty(B)$:

$$\{f_1, f_2\}_B(b) = -\Im \mathrm{tr}\left(b^{-1}(d^L f_1(b))b\, d^R f_2(b)\right)$$

where, for $f \in C^\infty(B)$, one defines $d^{L,R} f \in C^\infty(B, u(n))$ by

$$\frac{d}{ds}\Big|_{s=0} f(e^{sX} b e^{sY}) = \Im \mathrm{tr}\left(X d^L f(b) + Y d^R f(b)\right) \quad \forall X, Y \in \mathcal{B}$$

For any $\phi_1, \phi_2 \in C^\infty(U(n))$:

$$\{\phi_1, \phi_2\}_{U(n)}(g) = \mathrm{tr}\left(\mathbf{D}^R\phi_1(g) R^{\mathrm{i}}(\mathbf{D}^R\phi_2(g)) - \mathbf{D}^L\phi_1(g) R^{\mathrm{i}}(\mathbf{D}^L\phi_2(g))\right)$$

where, for $\phi \in C^\infty(U(n))$, one defines $\mathbf{D}^{L,R}\phi \in C^\infty(U(n), u(n))$ by

$$\frac{d}{ds}\Big|_{s=0} \phi(e^{sX} g e^{sY}) = \mathrm{tr}\left(X\mathbf{D}^L\phi(g) + Y\mathbf{D}^R\phi(g)\right) \quad \forall X, Y \in \mathcal{G}$$

Quasi-adjoint action

Following Lu [90]:

Poisson map from phase space into P-L group B is called (equivariant) *P-L moment map*. Every such map generates infinitesimal Poisson action of $U(n)$

$\Lambda_{L,R} : GL(n,\mathbb{C}) \to B$ moment maps generating left/right multiplications by $U(n)$.

The product $\Lambda := \Lambda_L \Lambda_R : GL(n,\mathbb{C}) \to B$ is also P-L moment map.
Λ generates infinitesimal 'quasi-adjoint' action of $U(n)$.
Concretely, for any $Y \in u(n)$ define vector field $\tilde{Y}$ on $GL(n,\mathbb{C})$ by

$$\mathcal{L}_{\tilde{Y}} f := \Im \mathrm{tr}\,(Y\{f,\Lambda\}_+ \Lambda^{-1}), \qquad \forall f \in C^\infty(GL(n,\mathbb{C}))$$

Integration of infinitesimal action yields $U(n)$ action on $GL(n,\mathbb{C})$:

$$\eta \triangleright K := \eta K \Xi_R(\eta \Lambda_L(K)), \qquad \eta \in U(n), \quad K \in GL(n,\mathbb{C})$$

Now can reduce $(GL(n,\mathbb{C}),\omega_+)$ by choosing $\nu \in B$ and imposing

moment map constraint: $\quad \Lambda(K) = \nu, \quad K \in GL(n,\mathbb{C})$.

But what dynamics to reduce, and **how to choose** ν?

The 'canonical free flows'

- First, flow of Hamiltonian $H = f \circ \Lambda_R$ with $f \in \mathbf{C^\infty(B)^c}$ is

$$K(t) = g_L(t) b_R^{-1}(t) = g_L(0) \exp\left[-t d^R f(b_R(0))\right] b_R^{-1}(0)$$

In other words, $b_R(t) = b_R(0)$ and $g_L(t) = g_L(0) \exp\left[-t d^R f(b_R(0))\right]$

Equivalently, $b_L(t) = b_L(0)$ and $g_R(t) = \exp\left[-t d^L f(b_L(0))\right] g_R(0)$

- Second, the flow of $\hat{H} = \phi \circ \Xi_R$ with $\phi \in \mathbf{C^\infty(U(n))^{U(n)}}$ reads

$$g_R(t) = \gamma(t) g_R(0) \gamma(t)^{-1}, \qquad b_L(t) = b_L(0)\beta(t)$$

with $\gamma(t) \in U(n)$, $\beta(t) \in B$ defined by $e^{\mathrm{i}t\mathbf{D}\phi(g_R(0))} = \beta(t)\gamma(t)$. Also

$$K(t)K^\dagger(t) = b_L(t) b_L(t)^\dagger = b_L(0) e^{2\mathrm{i}t\mathbf{D}\phi(g_R(0))} b_L(0)^\dagger$$

Solutions are obtained by Gram-Schmidt orthogonalization.

- **'Canonical free Hamiltonians' are invariant under the quasi-adjoint action of $U(n)$; thus can be reduced simultaneously.**

‘Unreduced Lax matrices’

generators of $C^\infty(B)^c$: $\quad f_k(b) := \frac{1}{2k}\mathrm{tr}\,(bb^\dagger)^k \quad \forall k \in \mathbb{Z}^*$
$/C^\infty(B)^c = C^\infty(B)^{U(n)}$ – dressing invariants/

$$\text{generators of } C^\infty(U(n))^{U(n)}: \quad \phi_k(g) := \frac{1}{2k}\mathrm{tr}\,(g^k + g^{-k})$$
$$\phi_{-k}(g) := \tfrac{1}{2k\mathrm{i}}\mathrm{tr}\,(g^k - g^{-k}) \quad \forall k \in \mathbb{Z}_+$$

Canonical Hamiltonians $H_k := f_k \circ \Lambda_R$ and $\hat{H}_k := \phi_k \circ \Xi_R$ are **spectral invariants** of matrix functions $\mathcal{L}$ and $\hat{\mathcal{L}}$ defined on the double by

$$\mathcal{L} := \Lambda_R \Lambda_R^\dagger \quad \text{and} \quad \hat{\mathcal{L}} := \Xi_R$$

We call $\mathcal{L}$ and $\hat{\mathcal{L}}$ unreduced Lax matrices.

The quasi-adjoint action operates on the ‘unreduced Lax matrices’ $\mathcal{L}$ and $\hat{\mathcal{L}}$ by similarity transformations. Hence $\mathcal{L}$ and $\hat{\mathcal{L}}$ yield Lax matrices for reduced systems obtained from $\{H_k\}$ and from $\{\hat{H}_k\}$.
We prove: $\mathcal{L}$ **and** $\hat{\mathcal{L}}$ **descend to the RS Lax matrices** L **and** $\hat{L}$**.**

Definition of the reduction

- First, fix value of moment map Λ to some constant $\nu \in B$.
- Second, factor level set $\Lambda^{-1}(\nu)$ by isotropy group G_ν of ν.

The crux is the choice $\nu := \nu(x)$: with $x \neq 0$ real parameter

$$\nu(x)_{kk} = 1, \quad \forall k, \quad \nu(x)_{kl} = (1 - e^{-2x})e^{(l-k)x}, \quad \forall k < l$$

Useful relation: $\nu(x)\nu(x)^\dagger = e^{-2x}\left[\mathbf{1}_n + \frac{e^{2nx} - 1}{n} v(x)v(x)^\dagger\right]$

with vector $v(x) \in \mathbb{R}^n$ defined by $v_k(x) = \sqrt{\frac{n(e^{2x}-1)}{1-e^{-2nx}}}e^{-kx}$

$F_{\nu(x)} := \Lambda^{-1}(\nu(x))$: **embedded** submanifold of $GL(n, \mathbb{C})$
$G_{v(x)} < U(n)$: isotropy group of $v(x)$ – **acts freely** on $F_{\nu(x)}$

Central $U(1) < U(n)$ acts trivially. $G_{v(x)} < G_{\nu(x)}$ isomorphic to $G_{\nu(x)}/U(1)$.
Reduced phase space is **smooth manifold** $F_{\nu(x)}/G_{v(x)}$.

Key facts about the reduced system

Consider natural embedding $\mathcal{E}$ and projection π

$$\mathcal{E} : F_{\nu(x)} \to D \equiv GL(n,\mathbb{C}), \quad \pi : F_{\nu(x)} \to F_{\nu(x)}/G_{v(x)} \equiv D_{\text{red}}$$

$(D_{\text{red}}, \omega_{\text{red}})$ is symplectic manifold characterized by $\mathcal{E}^*\omega_+ = \pi^*\omega_{\text{red}}$

$(D_{\text{red}}, \omega_{\text{red}})$ carries reduced canonical Hamiltonians defined by

$$\pi^* H_k^{\text{red}} = \mathcal{E}^* H_k, \qquad \pi^* \hat{H}_k^{\text{red}} = \mathcal{E}^* \hat{H}_k$$

$\{H_k^{\text{red}}\}$ and $\{\hat{H}_k^{\text{red}}\}$ form two **Abelian** algebras. Induce **complete flows** on D_{red}: obvious projections of 'canonical free flows'.

The aim is to exhibit concrete models of the reduced phase space. Any two models are symplectomorphic to each other naturally. If global sections of the principal $G_{v(x)}$ bundle $\pi : F_{\nu(x)} \to D_{\text{red}}$ exist, then they can be taken as models of $(D_{\text{red}}, \omega_{\text{red}})$.

We exhibit two models, which will be identified with (P, ω) and with the natural completion of $(\hat{P}, \hat{\omega})$. This explains Ruijsenaars' duality.

Preparation for describing the first model

Consider

$$T^*\mathbb{T}_n^0 \simeq \mathbb{T}_n^0 \times \mathbb{R}^n = \{(e^{2\mathrm{i}q}, p)\}, \qquad \Omega_{T^*\mathbb{T}_n^0} \equiv \sum_{k=1}^n dp_k \wedge dq_k$$

and the projection $\pi_1 : T^*\mathbb{T}_n^0 \to (T^*\mathbb{T}_n^0)/S_n \equiv T^*(\mathbb{T}_n^0/S_n) \equiv T^*Q(n)$, for which $\pi_1^*(\Omega_{T^*Q(n)}) = \Omega_{T^*\mathbb{T}_n^0}$. That is, consider S_n-covering of phase space $P = T^*Q(n)$.

Define the smooth map $\tilde{\mathcal{I}} : T^*\mathbb{T}_n^0 \to GL(n, \mathbb{C})$ by the following explicit formula:

$$\tilde{\mathcal{I}}(e^{2\mathrm{i}q}, p)_{kk} = e^{-p_k/2 - 2\mathrm{i}q_k} \prod_{m<k} \left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_m)}\right]^{-\frac{1}{4}} \prod_{m>k} \left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_m)}\right]^{\frac{1}{4}}$$

$$\tilde{\mathcal{I}}(e^{2\mathrm{i}q}, p)_{kl} = 0, \quad k > l, \qquad \tilde{\mathcal{I}}(e^{2\mathrm{i}q}, p)_{kl} = \tilde{\mathcal{I}}(e^{2\mathrm{i}q}, p)_{ll} \prod_{m=1}^{l-k} \frac{e^x e^{2\mathrm{i}q_l} - e^{-x} e^{2\mathrm{i}q_{k+m}}}{e^{2\mathrm{i}q_l} - e^{2\mathrm{i}q_{k+m-1}}} \quad k < l$$

Claim: the image of $T^*\mathbb{T}_n^0$ by $\tilde{\mathcal{I}}$ is a symplectic submanifold $\tilde{S} \subset F_{\nu(x)} \subset GL(n, \mathbb{C})$. $(\tilde{S}, \omega_+|_{\tilde{S}})$ and $T^*\mathbb{T}_n^0$ are symplectomorphic by $\tilde{\mathcal{I}}$, and furnish symplectic covering spaces of the reduced phase space.

The first model of the reduced phase space

With $\tilde{S} \subset \Lambda^{-1}(\nu(x)) \equiv F_{\nu(x)}$, the situation is summarized by the diagram:

$$\begin{array}{ccc} T^*\mathbb{T}_n^0 & \xrightarrow{\tilde{\mathcal{I}}} & \tilde{S} \subset F_{\nu(x)} \\ \pi_1 \downarrow & & \downarrow \pi \\ T^*Q(n) & \xrightarrow{\mathcal{I}} & \tilde{S}/S_n \simeq D_{\mathrm{red}} \end{array}$$

with induced S_n-action on $\tilde{S}$.

The map $\tilde{\mathcal{I}} : T^*\mathbb{T}_n^0 \to D$ is injective, its image lies in $F_{\nu(x)}$, and it verifies

$$\tilde{\mathcal{I}}^*\omega_+ = \Omega_{T^*\mathbb{T}_n^0}.$$

$\tilde{\mathcal{I}}$ descends to a diffeomorphism $\mathcal{I} : T^*Q(n) \to F_{\nu(x)}/G_{v(x)}$ defined by the equality

$$\mathcal{I} \circ \pi_1 = \pi \circ \tilde{\mathcal{I}},$$

and $\mathcal{I}$ satisfies $\mathcal{I}^*\omega_{\mathrm{red}} = \Omega_{T^*Q(n)}$, where $\pi : F_{\nu(x)} \to F_{\nu(x)}/G_{v(x)}$ is the projection.

Thus $(P, \omega) \equiv (T^*Q(n), \Omega_{T^*Q(n)})$ is a model of reduced phase space $(D_{\mathrm{red}}, \omega_{\mathrm{red}})$.

The composition $\mathcal{L} \circ \tilde{\mathcal{I}}$ gives (up to inessential similarity transformation) the Lax matrix L of the standard Ruijsenaars-Schneider system, where L is regarded as a function on the covering space $T^*\mathbb{T}_n^0$ of $P = T^*Q(n)$.

Hence trigo RS system (P, ω, L) is reduction of 'free' system $(D, \omega_+, \mathcal{L})$.

Preparations for the second model

Recall (incomplete) dual phase space, $\hat{P} = \mathbb{T}_n \times \mathfrak{C}_x = \{(e^{\mathrm{i}\hat{q}}, \hat{p})\}$ with $\hat{\omega} = d\hat{p}_k \wedge d\hat{q}_k$.

Consider $\hat{P}_c := \mathbb{C}^{n-1} \times \mathbb{C}^\times$ with the symplectic form

$$\hat{\omega}_c := \frac{\mathrm{i} dZ \wedge d\bar{Z}}{2\bar{Z}Z} + \mathrm{sign}(x) \sum_{j=1}^{n-1} \mathrm{i} dz_j \wedge d\bar{z}_j, \qquad Z \in \mathbb{C}^\times, \quad z \in \mathbb{C}^{n-1}.$$

Define the smooth injective map $\mathcal{Z}_x : \hat{P} \to \hat{P}_c$ by

$$z_j(x, \hat{q}, \hat{p}) = (\hat{p}_j - \hat{p}_{j+1} - |x|)^{\frac{1}{2}} \prod_{k=j+1}^{n} e^{-\mathrm{i}\hat{q}_k}, \quad Z(x, \hat{q}, \hat{p}) = e^{-\hat{p}_1} \prod_{k=1}^{n} e^{-\mathrm{i}\hat{q}_k}, \quad x > 0,$$

$$z_j(x, \hat{q}, \hat{p}) = (\hat{p}_j - \hat{p}_{j+1} - |x|)^{\frac{1}{2}} \prod_{k=1}^{j} e^{-\mathrm{i}\hat{q}_k}, \quad Z(x, \hat{q}, \hat{p}) = e^{-\hat{p}_n} \prod_{k=1}^{n} e^{-\mathrm{i}\hat{q}_k}, \quad x < 0.$$

$\mathcal{Z}_x$ is a symplectic embedding of $(\hat{P}, \hat{\omega})$ into $(\hat{P}_c, \hat{\omega}_c)$, $\mathcal{Z}_x^* \hat{\omega}_c = \hat{\omega}$.

The $\mathcal{Z}_x$-image $\hat{P}_c^0 := \mathcal{Z}_x(\hat{P}) \subset \hat{P}_c$ is dense open submanifold.
$\hat{P}_c \setminus \mathcal{Z}_x(\hat{P})$ consists of the points for which some z_j $(j = 1, \ldots, n-1)$ vanishes.

We construct smooth, injective map $k_x : \widehat{P} \to F_{\nu(x)}$ by explicit formula

$$k_x(e^{\mathrm{i}\widehat{q}}, \widehat{p}) := \left(\kappa_L(x)\aleph(x, e^{\mathrm{i}\widehat{q}})_{(x)}\zeta(x,\widehat{p})^{-1}\right) \triangleright \left(\theta(x,\widehat{p})e^{\mathrm{i}\widehat{q}}e^{\widehat{p}}\right)^{-1}$$

Here, with $\widehat{p} := \mathrm{diag}(\widehat{p}_1, \ldots, \widehat{p}_n)$, θ is $\mathrm{O}(n,\mathbb{R})$-valued function on the closure of $\mathfrak{C}_x$:

$$\theta(x,\widehat{p})_{jk} := \frac{\sinh(x)}{\sinh(\widehat{p}_k - \widehat{p}_j)} \prod_{m\neq j,k} \left[\frac{\sinh(\widehat{p}_j - \widehat{p}_m - x)\sinh(\widehat{p}_k - \widehat{p}_m + x)}{\sinh(\widehat{p}_j - \widehat{p}_m)\sinh(\widehat{p}_k - \widehat{p}_m)}\right]^{\frac{1}{2}}, \text{ if } j \neq k,$$

$$\theta(x,\widehat{p})_{jj} := \prod_{m\neq j} \left[\frac{\sinh(\widehat{p}_j - \widehat{p}_m - x)\sinh(\widehat{p}_j - \widehat{p}_m + x)}{\sinh^2(\widehat{p}_j - \widehat{p}_m)}\right]^{\frac{1}{2}}.$$

We also use $\mathrm{O}(n,\mathbb{R})$-valued functions $\kappa_L(x)$ and $\zeta(x,\widehat{p})$ and the diffeomorphism $\aleph : \mathbb{T}_n \to \mathbb{T}_n$ provided by

$$\aleph(x,\tau)_j := \prod_{k=j}^{n} \tau_k^{-1}, \quad x > 0, \qquad \aleph(x,\tau)_j := \prod_{k=1}^{j} \tau_k^{-1}, \quad x < 0,$$

and notation

$$\tau_{(x)} := \mathrm{diag}(\tau_2, \ldots, \tau_n, 1) \quad \text{if} \quad x > 0, \qquad \tau_{(x)} := \mathrm{diag}(1, \tau_1, \ldots, \tau_{n-1}) \quad \text{if} \quad x < 0.$$

The final result

• $\pi \circ k_x : \hat{P} \to D_{\text{red}}$ gives symplectic diffeomorphism onto open dense submanifold D^0_{red} of reduced phase space.

• $\hat{\mathcal{L}} \circ k_x$ gives (up to inessential similarity transformation) the dual Lax matrix $\hat{L}$.

• Thus $(\hat{P}, \hat{\omega}, \hat{L})$ represents the restriction on D^0_{red} of the reduction of the 'free' system $(D, \omega_+, \hat{\mathcal{L}})$.

• **The map $k_x \circ \mathcal{Z}_x^{-1} : \hat{P}^0_c \to F_{\nu(x)}$ extends uniquely to a smooth injective map $\hat{\mathcal{I}} : \hat{P}_c \to F_{\nu(x)}$ such that $\pi \circ \hat{\mathcal{I}} : \hat{P}_c \to D_{\text{red}}$ is a symplectic diffeomorphism. Therefore, $(\hat{P}_c, \hat{\omega}_c)$ is a model of the full reduced phase space.**

Ruijsenaars' restricted and global duality (action-angle) maps, $\mathcal{R}^0$ and $\mathcal{R}$, are obtained geometrically:

$$\begin{array}{ccccc} P^0 & \xrightarrow{\text{id}} & P^0 & \xrightarrow{\mathcal{I}^0} & F^0_{\nu(x)}/G_{v(x)} \\ \mathcal{R}^0 \downarrow & & \mathcal{R}^0_c \downarrow & & \downarrow \text{id} \\ \hat{P} & \xrightarrow{\mathcal{Z}_x} & \hat{P}^0_c & \xrightarrow{\pi\circ\hat{\mathcal{I}}^0} & F^0_{\nu(x)}/G_{v(x)} \end{array} \quad \text{and} \quad \begin{array}{ccc} P & \xrightarrow{\mathcal{I}} & F_{\nu(x)}/G_{v(x)} \\ \mathcal{R} \downarrow & & \downarrow \text{id} \\ \hat{P}_c & \xrightarrow{\pi\circ\hat{\mathcal{I}}} & F_{\nu(x)}/G_{v(x)} \end{array}$$

All $K \in F_{\nu(x)}$ satisfy $-\frac{1}{2}\log(KK^\dagger) \in \bar{\mathfrak{C}}_x$. Dense submanifold $F^0_{\nu(x)}$ is characterized by condition $-\frac{1}{2}\log(KK^\dagger) \in \mathfrak{C}_x$. $\hat{P}$ and P^0 are two models of $D^0_{\text{red}} \equiv F^0_{\nu(x)}/G_{v(x)}$.

Concluding remarks

Presented group theoretical method whereby obtains trigonometric Ruijsenaars-Schneider system and its **completed** dual in one stroke.

The idea was to follow natural generalization from ordinary to Poisson-Lie symmetry and reduce canonical free systems.

Technically simplifies parts of original work of Ruijsenaars [88,95].

Advantage: Complete flows and duality symplectomorphism result automatically.

Problems under investigation and plans for the future:

- Study compactified, hyperbolic and elliptic RS systems.
- Explore reduced systems at arbitrary moment map value.
- Quantum Hamiltonian reduction ($\sim$ works on special functions)

Etingof-Kirillov [94], Noumi [96]: Q.G. interpretation of Macdonald polynomials

- Connections to bispectrality and to separation of variables.
- Derive $BC(n)$ (van Diejen) systems in analogous manner.