Stability problems in convex and discrete geometry

Rolf Schneider

U Freiburg

Szeged, February 28, 2007

The idea of stability: Small changes cause only small effects.

Aim:
Explicit stability estimates for geometric uniqueness theorems.

A general (vague) principle:

Take any geometric uniqueness theorem. Suppose some condition C implies uniqueness.

If condition C is only "satisfied up to ϵ", does uniqueness hold "up to $f(\epsilon)$", with explicit f?

Surprisingly few proofs of classical uniqueness theorems yield this without problems.

An easy example: the isoperimetric problem in the plane

Among all planar convex domains of given area, precisely the circles have the smallest perimeter.

Bonnesen's inequality (1929) is the ideal type of a stability estimate:

$$L^2 - 4\pi A \geq 4\pi(R-r)^2$$

(L perimeter, A area, R circumradius, r inradius of a convex body $K \subset \mathbb{R}^2$).

Approximate equality immediately yields an estimate for the deviation from a circle:

$$L^2 - 4\pi A \leq \epsilon \quad \Rightarrow \quad R - r \leq \frac{1}{2\sqrt{\pi}}\sqrt{\epsilon}\,.$$

I know of no other case where a stability estimate is similarly easy and elegant.

Programme:

A guided tour, visiting some classical ($50 - 120$ years old) uniqueness theorems of convex geometry, where stability has been established (much later, or very recently), or not. We stop at:

Balls and ellipsoids

Simplices

Pairs of convex bodies

Discrete geometry

Balls and ellipsoids

(1) The isoperimetric inequality in $\mathbb{R}^d$

Among all d-dimensional convex bodies of given volume V, precisely the balls have smallest surface area S.

One of several possible stability estimates is given by

$$\left(\frac{S}{\sigma_d}\right)^d - \left(\frac{V}{\kappa_d}\right)^{d-1} \geq c(d) r^{d^2-3(d+1)/2} (R-r)^{\frac{d+3}{2}}.$$

(Groemer–R.S. 1991, further results by Osserman 1987 and Fuglede 1994).

The method of Groemer–R.S. (spherical harmonics, Aleksandrov-Fenchel inequalities) has the advantage that it yields similar stability results for inequalities between two intrinsic volumes (in those cases where balls are the only extremal bodies).

(2) The Liebmann-Süss theorem

Liebmann's theorem (1899) says that a smooth closed convex surface in $\mathbb{R}^3$ with constant Gauss curvature or constant mean curvature (soap bubbles) must be a sphere.

Are perturbed soap bubbles almost spherical?

Arnold 1993, improving R.S. 1990, showed: If $K \in \mathcal{K}^d$ is of class C^2 and its mean curvature H_1 satisfies

$$1 - \epsilon \leq H_1 \leq 1$$

for some $\epsilon \in [0, 1)$, then

$$\delta_2(K, B_K) \leq c(d)V(K)\sqrt{\epsilon}$$

for some unit ball B_K. Here δ_2 denotes the L_2-metric (the L_2-norm of the difference of the support functions).

The Liebmann-Süss theorem requires $H_r = 1$, for the rth elementary symmetric function of the principal curvatures. A smooth convex body with this property is a ball (Süss 1929).

Without smoothness:

The condition $H_r = 1$ is equivalent to $C_{d-r}(K,\cdot) = C_{d-1}(K,\cdot)$ (curvature measures), which makes sense for arbitrary convex bodies and characterizes balls (R.S. 1979).

Kohlmann 1996 was able to prove stability:

If $r \in \{2,\dots,d-1\}$, $\epsilon \geq 0$ is sufficiently small, and

$$(1-\epsilon)C_{d-1}(K,\cdot) \leq C_{d-r}(K,\cdot) \leq (1+\epsilon)C_{d-1}(K,\cdot),$$

then there is a ball B with

$$\delta(K,B) \leq c(d)\epsilon^{2/d(d+3)},$$

where δ is the Hausdorff metric.

Three characteristic properties of ellipsoids

(3) Brunn's characterization

The midpoint set $M(K, u)$ in direction u of the convex body K is the set of all midpoints of secants of K parallel to u.

If every midpoint set of K lies in a hyperplane, then K is an ellipsoid (Brunn).

Groemer 1994 has obtained a stability version: W.l.o.g., suppose that the Löwner ellipsoid $\mathcal{E}^K$ of K is the unit ball. If

$$V(M(K, u)) \leq \epsilon$$

for all u, then

$$\delta(K, \mathcal{E}^K) \leq c(d)\epsilon^{1/4}.$$

(4) Blaschke's characterization

Blaschke 1916: A convex body with planar shadow boundaries is an ellipsoid.

Precise formulation: Let $K \subset \mathbb{R}^d$ $(d \geq 3)$ be a convex body. Suppose to every line G through 0 there is a hyperplane H such that

$$K + G = (K \cap H) + G.$$

Then K is an ellipsoid.

Gruber 1997 proved: Let $\epsilon > 0$, and w.l.o.g suppose that the John ellipsoid $\mathcal{E}_K$ of K is the unit ball. If to every line G through 0 there is a slab P of thickness 2ϵ such that

$$K + G = (K \cap P) + G,$$

then

$$\delta(K, \mathcal{E}_K) \leq c(d)\epsilon^{1/4}.$$

(5) Monge's property

Let E be an ellipsoid. The vertices of all rectangular boxes circumscribed about E lie on a sphere (Monge).

This characterizes ellipsoids (Blaschke).

Burger–R.S. 1993 proved a stability version:

Let K be a convex body, let $\epsilon > 0$ be sufficiently small.

If the vertices of all rectangular boxes circumscribed about E lie between two concentric spheres of radii $(1-\epsilon)^{1/2}$ and $(1+\epsilon)^{1/2}$, then there exists an ellipsoid E with

$$\delta(K, E) \leq c\,\epsilon^{1/(d+1)}.$$

An open problem

The Blaschke–Santaló inequality (1917/1949) says that, for a convex body K with $K = -K$,

$$V(K)V(K^*) \leq \kappa_d^2,$$

where K^* denotes the polar body and κ_d is the volume of the d-dimensional unit ball.

Equality characterizes ellipsoids.

Prove a stability result.

Simplices

(6) The Minkowski measure of symmetry

The simplest measure of non-symmetry for convex bodies K is given by

$$q(K) := \min\{\lambda > 0 : \exists\, x \in K : -(K-x) \subseteq \lambda(K-x)\}.$$

It is known that $q(K) \leq d$, and equality characterizes simplices (Klee 1953).

Stability (Böröczky Jr. 2005, Guo 2005, R.S.): If $\epsilon > 0$ is sufficiently small, then (with T^d a simplex)

$$q(K) \geq d - \epsilon \quad \Rightarrow \quad d_{BM}(K, T^d) < 1 + 4d\epsilon,$$

where d_{BM} is the Banach–Mazur distance, defined by

$$d_{BM}(K, L) := \min\{\lambda \geq 1 : \exists\, A \in \mathrm{Aff}(d)\ \exists\, x \in \mathbb{R}^d : L \subseteq AK \subseteq \lambda L + x\}.$$

(7) The difference body inequality, or Rogers–Shephard inequality

The difference body of the convex body is the body

$$\mathsf{D}K := K - K = \{x - y : x, y \in K\}.$$

The Rogers–Shephard inequality (1957) says that

$$\frac{V(\mathsf{D}K)}{V(K)} \leq \binom{2d}{d},$$

and equality characterizes simplices.

Böröczky Jr. 2005 proved a stability result: If $\epsilon > 0$, then

$$\frac{V(\mathsf{D}K)}{V(K)} \geq (1-\epsilon)\binom{2d}{d} \quad \Rightarrow \quad d_{BM}(K, T^d) \leq 1 + c(d)\epsilon\,.$$

An open problem

Recall the definition of the Banach–Mazur distance:

$d_{BM}(K, L) := \min\{\lambda \geq 1 : \exists\, A \in \mathsf{Aff}(d)\ \exists\, x \in \mathbb{R}^d : L \subseteq AK \subseteq \lambda L + x\}.$

If B^d denotes a ball, then

$$d_{BM}(K, B^d) \leq d,$$

(John 1948) and equality characterizes simplices (Leichtweiss 1959, Palmon 1992).

Prove a stability result.

(8) A weaker inequality

Let $\mathcal{E}_K$ be the John ellipsoid of K (the ellipsoid of largest volume contained in K), and let $\mathcal{E}^K$ be the Loewner ellipsoid of K (the ellipsoid of smallest volume containing K). Put

$$vq(K) := \left(\frac{V(\mathcal{E}^K)}{V(\mathcal{E}_K)}\right)^{1/d}.$$

Then

$$vq(K) \leq d,$$

with equality if and only if K is a simplex.

Hug–R.S. 2007 proved: If $\epsilon > 0$ is sufficiently small, then

$$vq(K) \geq (1-\epsilon)d \quad \Rightarrow \quad d_{BM}(K, T^d) \leq 1 + c(d)\epsilon^{1/4}.$$

(9) Simplices contained in a fixed ball B

(a) A simplex of largest volume in B is regular.

Stability (Hug–R.S. 2004): Let T^d be a regular simplex inscribed to B, and let ϑ be a suitable measure of deviation of a simplex from the class of regular simplices inscribed to B. Then

$$\vartheta(S) \geq \epsilon \;\; \Rightarrow \;\; V(S) \leq (1 - c(d)\,\epsilon^2)\,V(T^d).$$

(b) A simplex of largest surface area in B is regular (R.M. Tanner 1974).

Stability: unknown

(c) A simplex of largest mean width in B is conjectured to be regular. Unknown for $d > 3$.

Pairs of convex bodies

Many uniqueness theorems for convex bodies characterize pairs of homothetic bodies or pairs of translates of a convex body.

Examples: In the Brunn–Minkowski inequality (1887)

$$V((1-\lambda)K+\lambda L)^{1/d} \geq (1-\lambda)V(K)^{1/d} + \lambda V(L)^{1/d},$$

and also in Minkowski's inequality (1903) for mixed volumes,

$$V(K,L,\dots,L)^d \geq V(K)V(L)^{d-1},$$

equality for d-dimensional convex bodies holds if and only if K and L are homothetic.

Corresponding stability estimates are due to Diskant 1973, 1989 and Groemer 1988.

(10) The Aleksandrov–Fenchel–Jessen theorem

If two smooth closed convex hypersurfaces have, at points with the same outer normal vectors, the same ith elementary symmetric function of the principal radii of curvature (for some $i \in \{1, \dots, d-1\}$), then they are translates of each other (Chern 1959).

Aleksandrov 1937 and Fenchel–Jessen 1938 proved this without smoothness, using the surface area measure $S_i(K, \cdot)$ of order i.

A stability estimate (R.S. 1989): Let K, L be convex bodies with $rB^d \subset K, L \subset RB^d$, where $0 < r < R$ are given, let $i \in \{1, \dots, d-1\}$ and $0 \le \epsilon < \epsilon_0$. If

$$|S_i(K, \cdot) - S_i(L, \cdot)| \le \epsilon$$

then

$$\delta(K, L + t) \le c\,\epsilon^q \qquad \text{with } q = \frac{1}{(d+1)2^{i-1}},$$

with suitable t, where c depends on d, r, R, ϵ_0.

(11) Aleksandrov's projection theorem

For $K \in \mathcal{K}^d$ and $u \in S^{d-1}$, let $V_{d-1}(K,u)$ denote the $(d-1)$-dimensional volume of the orthogonal projection of K to the hyperplane $u^\perp$ through 0 orthogonal to u.

Aleksandrov's projection theorem (1937) says that if $K, L \in \mathcal{K}^d$ are d-dimensional and centrally symmetric with respect to 0, and satisfy

$$V_{d-1}(K,u) = V_{d-1}(L,u) \qquad \text{for all } u \in S^{d-1},$$

then $K = L$.

Stability result of Bourgain–Lindenstrauss 1988:

$$\delta(K,L) \le c\|V_{d-1}(K,\cdot) - V_{d-1}(L,\cdot)\|_2^q, \qquad 0 < q < \frac{2}{d(d+4)}.$$

Kiderlen 2007 improved the exponent ($d+1$ instead of $d+4$).

There are counterparts, with the $(d-1)$-volume replaced by the mean width (Goodey–Groemer 1990), or by the kth intrinsic volume, $k \in \{2, \dots, d-2\}$ (Hug–R.S. 2002).

But central symmetry is always a necessary assumption.

Projection data suitable for the determination of non-symmetric convex bodies were investigated by Groemer 1997 ('semi-girth') and Goodey–Weil 2006 ('directed projection functions').

Similar investigations, by the same authors, concern the determination of non-symmetric bodies by data of sections with hyperplanes through a fixed interior point.

We propose particularly elementary projection and section data:

(12) Determination of non-symmetric bodies from projections

The mean width and the Steiner point (centroid of the Gauss curvature) of K can be defined via the support function h_K, by

$$M(K) = \frac{2}{d\kappa_d} \int_{S^{d-1}} h_K \, \mathrm{d}\sigma, \qquad s(K) = \frac{1}{\kappa_d} \int_{S^{d-1}} h_K(u) u \, \mathrm{d}\sigma(u).$$

Let $M'(K,u)$ and $s(K,u)$ denote, respectively, the mean width and the Steiner point of the orthogonal projection of K to $u^\perp$.

A stability result (R.S. 2007):

$$\|M'(K,\cdot) - M'(L,\cdot)\|_2 \le \epsilon \quad \text{and} \quad \|s(K,\cdot) - s(L,\cdot)\|_2 \le \epsilon$$

together imply

$$\delta_2(K,L) \le c\,\epsilon^{2/d}.$$

(13) Determination of non-symmetric bodies from sections

Let K, L be convex bodies containing 0 as interior point. Let $V_{d-1}(K,u)$ and $c_{d-1}(K,u)$ denote, respectively, the $(d-1)$-volume and the centroid of the section of K by $u^\perp$.

A stability result (Böröczky Jr.–R.S. 2007): If $rB^d \subset K, L \subset RB^d$ and $0 \leq \epsilon < \epsilon_0$, then

$$\|V_{d-1}(K,\cdot) - V_{d-1}(L,\cdot)\|_2 \leq \epsilon \quad \text{and} \quad \|c_{d-1}(K,\cdot) - c_{d-1}(L,\cdot)\|_2 \leq \epsilon$$

together imply

$$\rho_2(K,L) \leq c\,\epsilon^{2/d}.$$

Here $\rho_2(K,L)$ denotes the radial L_2-distance of K and L (the L_2-norm of the difference of the radial functions). The constant c depends on d, r, R, ϵ_0.

Open problems

The preceding two results rely on stability estimates for the spherical Radon transform, obtained by spherical harmonics.

The obvious generalizations of the preceding results are open. For example:

Is a convex body uniquely determined by the $(d-1)$-volumes and the centroids of its orthogonal projections on hyperplanes?

If yes, what about stability?

Discrete Geometry

(14) Finite ball packings

Böröczky Jr. 1994 has used stability estimates from no. (1) to prove results of the following type:

Let Q be the convex hull of a packing of n unit balls in $\mathbb{R}^d$. Let r and R denote inradius and circumradius, respectively.

If Q has minimal surface area, then

$$\frac{r(Q)}{R(Q)} \geq 1 - \frac{c(d)}{n^{2/(d+3)}}.$$

The surface area can be replaced by the kth intrinsic volume, $k \in \{1, \ldots, d-1\}$, but <u>not</u> by the volume!

(15) Thinnest circle coverings

The density ϑ of a covering of $\mathbb{R}^2$ by unit circles satisfies

$$\vartheta \geq \frac{2\pi}{\sqrt{27}}\,.$$

Equality holds if the centers of the circles form a regular hexagonal lattice.

A stronger finite covering result due to L. Fejes Tóth 1953 was improved by Gruber 1997 to a stability result:

Let $C \subset \mathbb{R}^2$ be a finite set, let $\sigma, \delta > 0$. The point $c \in C$ is called center of a (σ, δ)-regular hexagon in C, if there exist points $c_1, \ldots, c_6 \in C$ with

$$\{x \in C : \|x - c\| \leq 1.5\,\sigma\} = \{c, c_1, \ldots, c_6\},$$

$$(1 - \delta)\sigma \leq \|c - c_k\|, \|c_{k+1} - c_k\| \leq (1 + \delta)\sigma.$$

Let H be a convex polygon with at most 6 vertices; let $\epsilon > 0$ be sufficiently small. Consider a covering of H by n congruent (sufficiently small)) circles with density

$$\vartheta < \frac{2\pi}{\sqrt{27}}(1+\epsilon).$$

Let C be the set of centers of the circles. With a suitable number $\sigma > 0$, the following holds:

Each point of C, with at most $50\epsilon^{1/3}n$ exceptions, is the center of a $(\sigma, 500\epsilon^{1/3})$-regular hexagon.

Consequence. Let $J \subset \mathbb{R}^2$ be a Jordan measurable set with positive measure. For $n \in \mathbb{N}$, consider a covering of J by n congruent circles such that the densities of these coverings tend to $2\pi/\sqrt{27}$ as $n \to \infty$. Then, as $n \to \infty$, the set of centers of the nth covering is asymptotically a regular hexagonal pattern.

(16) Random mosaics

Let X be a discrete point set in $\mathbb{R}^d$. The Voronoi cell of $p \in X$ is

$$C(p, X) := \{q \in \mathbb{R}^d : \|q - p\| \leq \|q - x\| \text{ for all } x \in X\},$$

and the system

$$V_X := \{C(p, X) : p \in X\}$$

is the Voronoi mosaic induced by X.

Now let X be random, especially a homogeneous Poisson process of intensity 1 (the number of points of X in a measurable set A has a Poisson distribution, with parameter equal to the Lebesgue measure of A). What we obtain is called the Poisson–Voronoi mosaic V_X.

Its typical cell (a well-defined kind of average cell) is stochastically equivalent to the random polytope

$$Z = C(0, X \cup \{0\}).$$

It seems plausible that typical cells of large volume are approximately balls (a variant of D.G Kendall's conjecture). This can be verified, but not easily.

One tool is a stability result for an inequality of isoperimetric type.

More generally, we measure the size of Z by its kth intrinsic volume V_k ($k \in \{1, \dots, d\}$).
We measure the deviation from a ball with center 0 by

$$\vartheta(Z) := \frac{R_0 - r_0}{R_0 + r_0},$$

where R_0 (r_0) is the radius of the smallest ball with center 0 containing Z (the largest ball with center 0 contained in Z).

Hug–Reitzner–R.S. 2004 proved that, for given $\epsilon > 0$ and all $a > 0$,

$$\mathbb{P}(\vartheta(Z) \geq \epsilon \mid V_k(Z) \geq a) \leq c \exp\left[-c_0\, \epsilon^{(d+3)/2}\, a^{d/k}\right].$$

This, too, is stability estimate: for the deviation from the asymptotic spherical shape, if the kth intrinsic volume is sufficiently large.

The proof uses the functional Φ defined by

$$\Phi(K) := \frac{1}{d}\int_{S^{d-1}} h_K(u)^d\, \sigma(\mathrm{d}u)$$

and a stability estimate for the inequality of isoperimetric type

$$\Phi(K) \geq \kappa_d^{1-d/k}\, V_k(K)^{d/k}.$$

The starting point was the problem of D.G. Kendall (1940s/1987) about the shape of large zero cells in planar Poisson line mosaics with rigid motion invariant distribution.

In joint work with D. Hug and M. Reitzner, various higher dimensional generalizations have been studied. A very general version (Hug–R.S 2007) concerns the zero cell Z_0 of a Poisson hyperplane mosaic (neither stationary nor isotropic) in $\mathbb{R}^d$. Probability estimates of the form

$$\mathbb{P}(\vartheta(Z_0) \geq \epsilon \mid \Sigma(Z_0) \geq a) \leq c \ \exp\left[-c_0\, f(\epsilon)\, a^{r/k}\right]$$

are derived from geometric stability estimates of the type

$$\vartheta(K) \geq \epsilon \ \Rightarrow \ \Phi(K) \geq (1 + f(\epsilon))\tau\Sigma(K)^{r/k}$$

(Σ an axiomatically defined size functional, Φ und r determinded by the distribution of the hyperplane process).